Funkcija dvije nezavisne promjenjive

Neka je S neprazan podskup prostora $\mathbb{R}^2$ i $T \subseteq \mathbb{R}$. Ako svakoj tački $M(x,y) \in S$ možemo unaprijed po datom pravilu f pridružiti jednu i samo jednu realnu vrijednost $z \in T$, tada kažemo da je data realna f-ja dvije realne promjenjive f iz $\mathbb{R}^2$ u $\mathbb{R}$ (sa skupa $S \subseteq \mathbb{R}$ u skup $T \subseteq \mathbb{R}$) i pišemo $z = f(x,y)$. Skup S na kojem je određena f-ja f naziva se domen ili definiciono područje f-je f (označavat ćemo ga sa $\mathcal{D}(f)$), a skup $f(A)$ skup vrijednosti f-je f ili kodomen (označavat ćemo ga sa $\mathcal{R}(f)$. Ako za f-ju, zadanu analitički (formulom) nije data oblast njene definisanosti, onda se pod njom podrazumjeva skup svih tačaka $M \in \mathbb{R}^2$ u kojoj f-ja, odnosno njen analitički izraz imaju određenu realnu vrijednost.

(#) Za svaku od sljedećih f-ja, izračunati $f(3,2)$, i odrediti i skicirati domen.

a) $f(x,y) = \dfrac{\sqrt{x+y+1}}{x-1}$ b) $f(x,y) = x \ln(y^2 - x)$

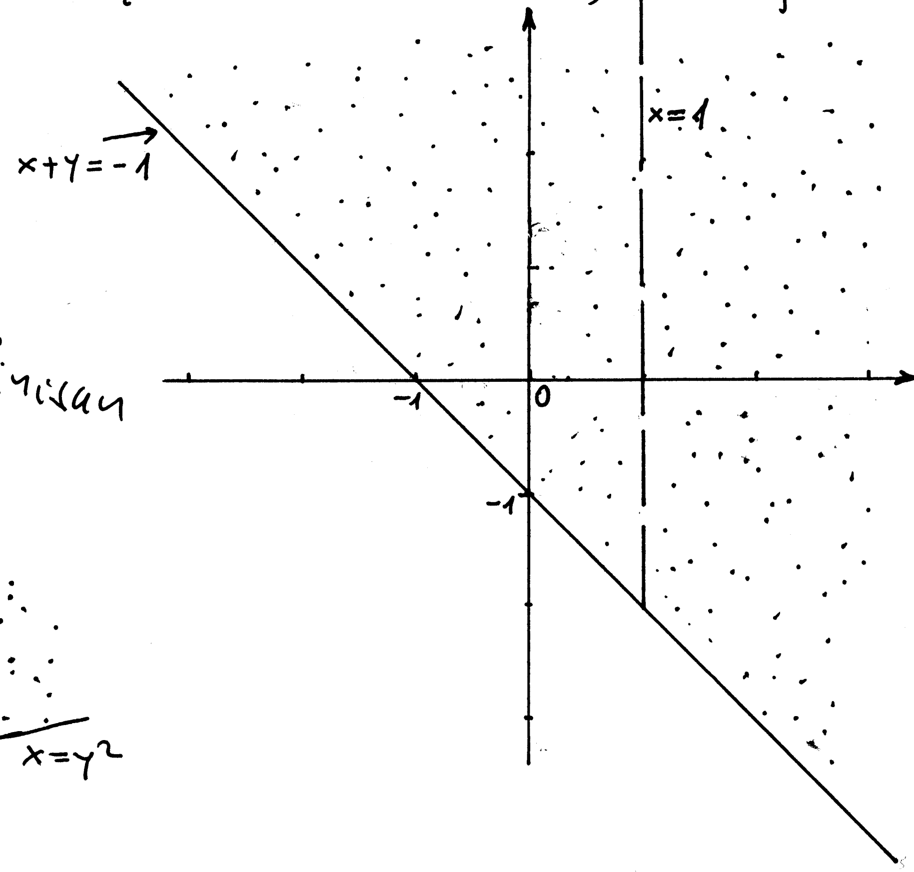
Rj.

a) $f(3,2) = \dfrac{\sqrt{3+2+1}}{3-1} = \dfrac{\sqrt{6}}{2}$

Izraz za f-ju $f(x,y)$ ima smisla ako je nazivnik različit od nule i ako je vrijednost pod korijenom nenegativna:

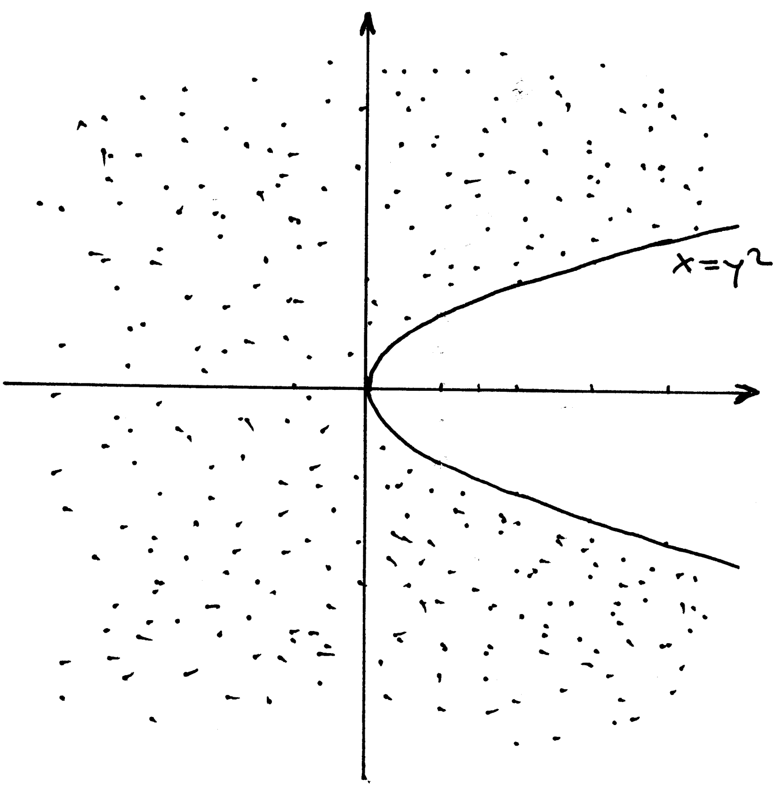
$$\begin{array}{l} x-1 \neq 0 \\ x+y+1 \geq 0 \end{array} \Rightarrow \begin{array}{l} x \neq 1 \\ x+y \geq -1 \end{array}$$

Domen f-je f je $D = \{(x,y) \in \mathbb{R}^2 \mid x+y \geq -1,\ x \neq 1\}$



b) $f(3,2) = 3\ln(2^2-3)$
$= 3\ln(4-3) = 3\underbrace{\ln 1}_{=0} =$
$= 0$

Izraz $\ln(y^2-x)$ je definisan samo ako je $y^2 - x > 0$



$D = \{(x,y) \mid x < y^2\}$

(#) Odrediti domen i rang f-je $g(x,y)=\sqrt{9-x^2-y^2}$

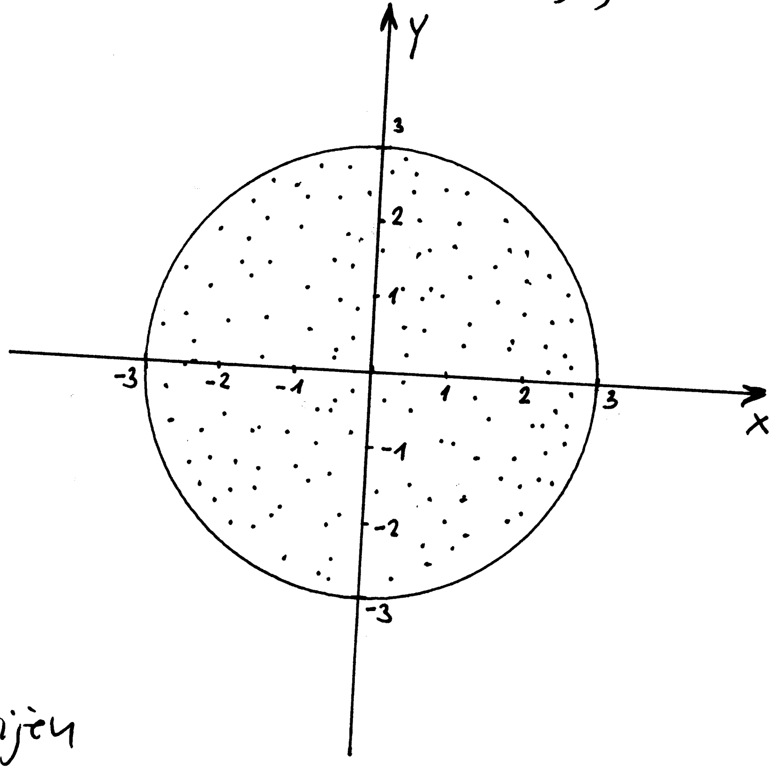
Rj.

F-ja ima smisla akko $9-x^2-y^2 \geq 0$

$$x^2+y^2 \leq 9$$

Domen f-je $g(x,y)$ je $D=\{(x,y)\in \mathbb{R}^2 \mid x^2+y^2 \leq 9\}$

(znamo da je $x^2+y^2=9$ krug sa centrom u tački $C(0,0)$ poluprečnika $r=3$).



Rang f-je g je

$$\{z\in\mathbb{R} \mid z=\sqrt{9-x^2-y^2},\ (x,y)\in D\}$$

Primjetimo da je

$$9-x^2-y^2 \leq 9 \quad \text{za } \forall (x,y)\in D$$

pa je $\sqrt{9-x^2-y^2} \leq 3$

z je pozitivan kvadratni korijen

$$z \geq 0$$

Prema tome, rang f-je $g(x,y)$ je

$$\{z \mid 0 \leq z \leq 3\} = [0,3]$$

(#) Skicirati graf f-je $f(x,y) = 6 - 3x - 2y$.

Rj.

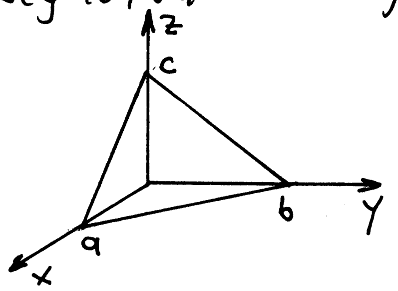
Graf f-je $f(x,y)$ ima jednačinu $z = 6 - 3x - 2y$

$$3x + 2y + z = 6$$

ovo predstavlja ravan.

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

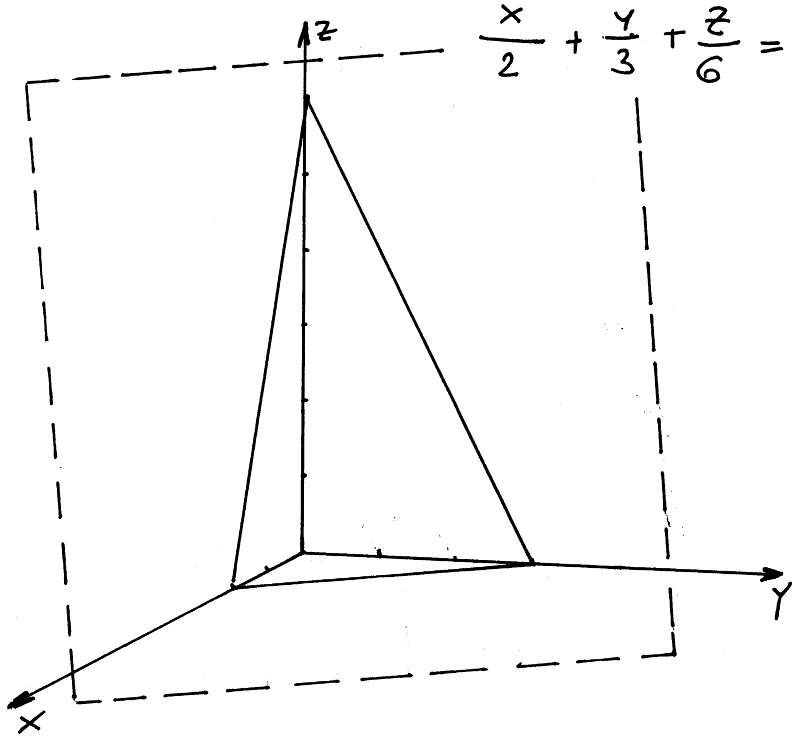
segmentni oblik jednačine ravni



U našem slučaju

$$3x + 2y + z = 6 \quad /:6$$

$$\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$



(#) Skicirati graf f-je $g(x,y)=\sqrt{9-x^2-y^2}$.

Rj. Graf f-je ima jednačinu $z=\sqrt{9-x^2-y^2}$

$$z=\sqrt{9-x^2-y^2} \quad /^2$$

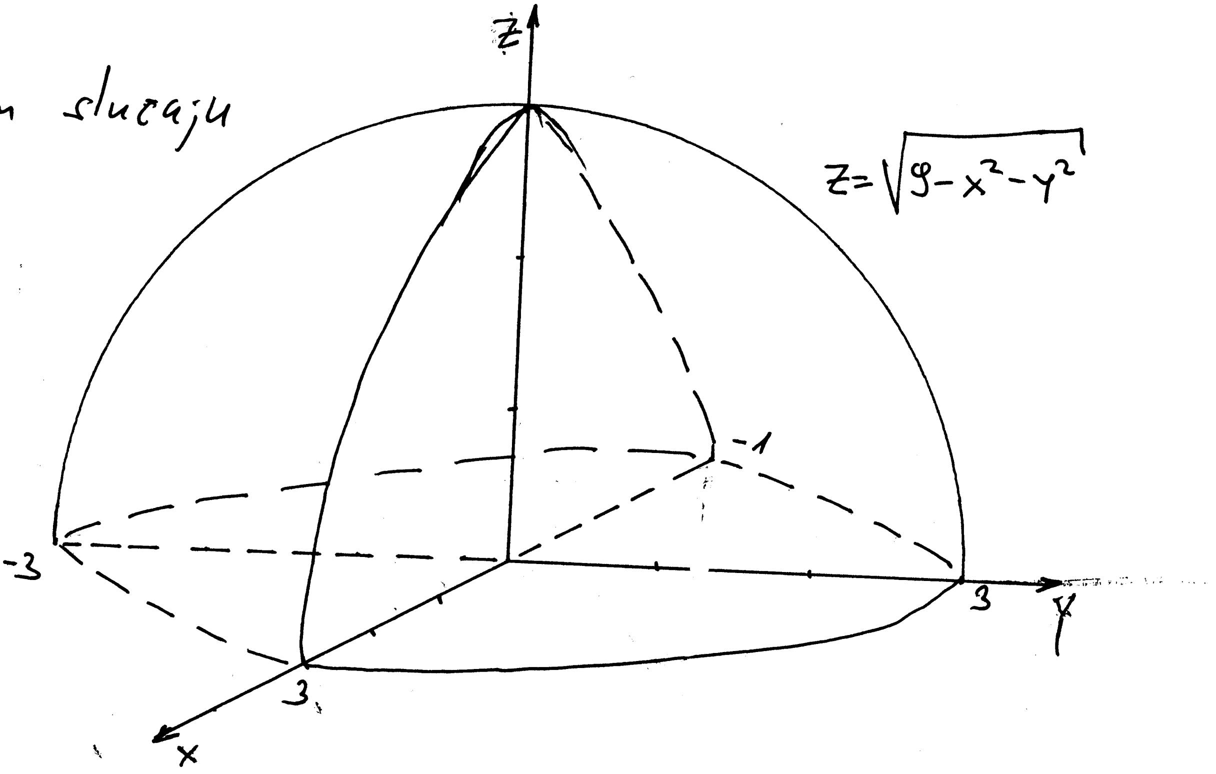
$$z^2=9-x^2-y^2$$

$$x^2+y^2+z^2=9$$

$x^2+y^2+z^2=R^2$

je jednačina sfere sa centrom u koordinatnom početku poluprečnika R

U našem slučaju



Zadaci za vježbu

§ 2. Početno proučavanje funkcije

Oblast definisanosti

2975. Oblast koja leži unutar paralelograma, obrazovanog pravama: $y=0$, $y=2$, $y=\frac{1}{2}x$, $y=\frac{1}{2}x-1$ prikazati pomoću nejednakosti.

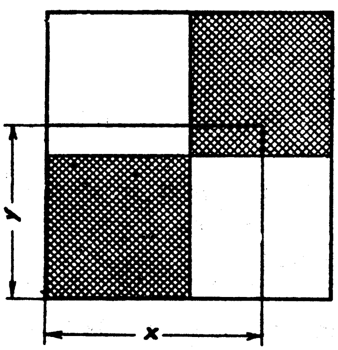
2976. Oblast ograničenu parabolama $y=x^2$ i $x=y^2$ (uključujući granice) definisati nejednakostima.

2977. Opisati pomoću nejednakosti otvorenu oblast, ograničenu jednakostraničnim trouglom stranice a, sa jednim temenom u koordinatnom početku, drugim — na pozitivnom delu x-ose, i trećim — u prvom kvadrantu.

2978. Oblast je ograničena beskonačnim kružnim cilindrom poluprečnika R (isključujući granice), čija je osa paralelna z-osi i prolazi kroz tačku (a, b, c); opisati ovu oblast pomoću nejednakosti.

2979. Oblast ograničenu sferom poluprečnika R sa centrom u tački (a, b, c) (uključujući granicu) definisati pomoću nejednakosti.

2980. Temena pravouglog trougla leže unutar kruga poluprečnika R. Površina S trougla je funkcija njegovih kateta x i y: $S=\varphi(x, y)$: naći: a) oblast definisanosti funkcije φ; b) oblast definisanosti odgovarajućeg analitičkog izraza.



Sl. 57

2981. U loptu poluprečnika R upisana je prava piramida sa pravougaonikom u osnovi. Zapremina V piramide je funkcija osnovnih ivica x i y. Hoće li ova funkcija biti jednoznačno definisana? Sastaviti njoj odgovarajući analitički izraz. i naći oblast definisanosti funkcije i pomenutog analitičkog izraza.

2982. Kvadratna daska se sastoji iz četiri kvadratna polja, dva crna i dva bela kao što je to prikazano na sl. 57; stranica svakog od njih ima dužinu 1. Uočimo pravougaonik čije su stranice x i y paralelne stranicama daske i čiji se jedan ugao poklapa sa njenim crnim uglom. Površina crnog dela ovog pravougaonika biće funkcija od x i y. Naći oblast definisanosti ove funkcije. Izraziti ovu funkciju analitički.

U zadacima 2983—3002 naći oblast definisanosti datih funkcija

2983. $z=\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$.

2984. $z=\ln(y^2-4x+8)$.

2985. $z=\frac{1}{R^2-x^2-y^2}$.

2986. $z=\sqrt{x+y}+\sqrt{x-y}$.

2987. $z=\frac{1}{\sqrt{x+y}}+\frac{1}{\sqrt{x-y}}$.

2988. $z=\arcsin\frac{y-1}{x}$.

2989. $z=\ln xy$.

2990. $z=\sqrt{x-\sqrt{y}}$.

2991. $z=\arcsin\frac{x^2+y^2}{4}+\operatorname{arcsec}(x^2+y^2)$.

2992. $z=\frac{\sqrt{4x-y^2}}{\ln(1-x^2-y^2)}$.

2993. $z=\sqrt{\frac{x^2+2x+y^2}{x^2-2x+y^2}}$.

2994. $z = xy + \sqrt{\ln \frac{R^2}{x^2+y^2}} + \sqrt{x^2+y^2-R^2}$.

2995. $z = \operatorname{ctg} \pi (x+y)$.

2996. $z = \sqrt{\sin \pi (x^2+y^2)}$.

2997. $z = \sqrt{x \sin y}$.

2998. $z = \ln x - \ln \sin y$.

2999. $z = \ln [x \ln (y-x)]$.

3000. $z = \arcsin [2y(1+x^2)-1]$.

Rješenja

2975. $0<y<2$; $-1<y-\frac{1}{2}x<0$. **2976.** $x^2 \leqslant y \leqslant \sqrt{x}$.

2977. $0<y<x\sqrt{3}$; $y<(a-x)\sqrt{3}$. **2978.** $(x-a)^2+(y-b)^2<R^2$; $-\infty<z<\infty$.

2979. $(x-a)^2+(y-b)^2+(z-c)^2 \leqslant R^2$. **2980.** a) $x^2+y^2 \leqslant 4R^2$; b) $-\infty<x<\infty$; $-\infty<y<\infty$.

2981. $v=\frac{1}{6}xy(2R \pm \sqrt{4R^2-x^2-y^2})$; funkcija nije jednoznačna. Oblast definisanosti funkcije je $x^2+y^2 \leqslant 4R^2$; $x>0$, $y>0$. Oblast definisanosti analitičkog izraza je $x^2+y^2 \leqslant 4R^2$.

2982. Za $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$ $S=xy$; za $1 \leqslant x \leqslant 2$, $2 \leqslant y$ $S=x$;
za $0 \leqslant x \leqslant 1$, $1 \leqslant y$ $S=x$; za $2 \leqslant x$, $1 \leqslant y \leqslant 2$ $S=y$;
za $1 \leqslant x$ $0 \leqslant y \leqslant 1$ $S=y$; za $2 \leqslant x$, $2 \leqslant y$ $S=2$;
za $1 \leqslant x \leqslant 2$, $1 \leqslant y \leqslant 2$ $S=xy-x-y+2$; funkcija nije definisana za $x<0$ i $y<0$.

2983. $\frac{x^2}{a^2}+\frac{y^2}{b^2} \leqslant 1$. **2984.** $y^2>4x-8$.

2985. Sva ravan izuzev tačaka kružne linije $x^2+y^2=R^2$.

2986. Unutrašnjost desnog pravog ugla koji obrazuju simetrale koordinatnih uglova, uključujući i odgovarajuće delove simetrale, tj.

$$x+y \geqslant 0, \quad x-y \geqslant 0.$$

2987. Ista kao i u zad. 2986, samo bez tačaka na granici oblasti.

2988. Unutrašnjost desnog i levog ugla koje obrazuju prave $y=1+x$ i $y=1-x$, uključujući i te prave, ali bez njihove presečne tačke:

$1-x \leqslant y \leqslant 1+x$ $(x>0)$, $1+x \leqslant y \leqslant 1-x$ $(x<0)$. (za $x=0$ funkcija nije definisana).

2989. Unutrašnjost prvog i trećeg kvadranta.

2990. Zatvorena oblast između pozitivnog dela apscisne ose i parabole $y=x^2$ (isključujući i granicu):

$$x \geqslant 0, \ y \geqslant 0; \ x^2 \geqslant y.$$

2991. Prstenasta oblast između krugova $x^2+y^2=1$ i $x^2+y^2=4$, uključujući i samo krugove: $1 \leqslant x^2+y^2<4$.

2992. Deo ravni koji leži unutar parabole $y^2=4x$, između parabole i kruga $x^2+y^2=1$, uključujući luk parabole izuzev njegovog temena, i isključujući luk kruga.

2993. Deo ravni koji leži izvan krugova čiji su poluprečnici jednaki jedinici a centri su im u tačkama $(-1, 0)$ i $(1, 0)$; tačke prvog kruga pripadaju oblasti, tačke drugog ne pripadaju.

2994. Samo tačke kružne linije $x^2+y^2=R^2$.

2995. Sva ravan, izuzev pravih $x+y=n$ (n je ma koji ceo broj, pozitivan, negativan ili nula).

2996. Unutrašnjost kruga $x^2+y^2=1$ i prsten $2n \leqslant x^2+y^2 \leqslant 2n+1$ (n je ceo broj), uključujući i granice.

2997. Ako je $x \geqslant 0$, onda je $2n\pi \leqslant y \leqslant (2n+1)\pi$, ako je $x<0$, onda je $(2n+1)\pi \leqslant y \leqslant (2n+2)\pi$, pri čemu je n ceo broj.

2998. $x>0$; $2n\pi<y<2(n+1)\pi$ (n je ceo broj).

2999. Otvorena šrafirana oblast prikazana na sl. 83: za $x>0$ je $y>x+1$; za $x<0$ je $x<y<x+1$.

Parcijalni izvodi f-ja više promjenjivih

Posmatrajmo f-ju z dvije promjenjive $z = f(x, y)$.
Parcijalni izvod po x-u označavamo sa z'_x ili sa $\frac{\partial z}{\partial x}$ (delta z po delta x) ili sa f'_x i definišemo

$$z'_x = \frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x}$$

Parcijalni izvod po y-nu označavamo sa z'_y ili sa $\frac{\partial z}{\partial y}$ (∂ - delta) ili sa f'_y i definišemo

$$z'_y = \frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

(#) Odrediti parcijalne izvode f-ja

a) $z = x^3 + 5xy^2 - y^3$

b) $u = \frac{x}{y} + \frac{y}{z} - \frac{z}{x}$

c) $v = \sqrt[x]{e^y}$

Rj. a) Kad vadimo izvod po x-u, samo x tumačimo kao promjenjivu, sve ostalo tumačimo kao broj.

$$\frac{\partial z}{\partial x} = 3x^2 + 5y^2 .$$

Analogno za y-on $\quad \frac{\partial z}{\partial y} = 10xy - 3y^2 .$

b)
$$\frac{\partial u}{\partial x} = \frac{1}{y} - z \cdot \left(\frac{1}{x}\right)'_x = \frac{1}{y} - z \cdot (-1)x^{-2} = \frac{1}{y} + \frac{z}{x^2}$$

$$\frac{\partial u}{\partial y} = x \cdot (-1)y^{-2} + \frac{1}{z} = -\frac{x}{y^2} + \frac{1}{z}$$

$$\frac{\partial u}{\partial z} = y \cdot \left(\frac{1}{z}\right)'_z - \frac{1}{x} = y \cdot (-1) z^{-2} - \frac{1}{x} = -\frac{y}{z^2} - \frac{1}{x}$$

c)
$$\frac{\partial v}{\partial x} = \left(e^{\frac{y}{x}}\right)'_x = e^{\frac{y}{x}} \cdot \left(\frac{y}{x}\right)'_x = y e^{\frac{y}{x}} \cdot (x^{-1})'_x = -y e^{\frac{y}{x}} \cdot x^{-2} = -\frac{y}{x^2} e^{\frac{y}{x}}$$

$$\frac{\partial v}{\partial y} = \left(e^{\frac{y}{x}}\right)'_y = e^{\frac{y}{x}} \cdot \left(\frac{y}{x}\right)'_y = \frac{1}{x} e^{\frac{y}{x}}$$

(#) Pronaći vrijednost parcijalnih izvoda datih f-ja u datim tačkama

a) $f(\alpha, \beta) = \cos(m\alpha - n\beta)$, $\alpha = \frac{\pi}{2m}$, $\beta = 0$;

b) $z = \ln(x^2 - y^2)$, $x = 2$, $y = -1$.

Rj: a) $f'_\alpha = -\sin(m\alpha - n\beta)\cdot(m\alpha - n\beta)'_\alpha = -m\sin(m\alpha - n\beta)$

$f'_\beta = -\sin(m\alpha - n\beta)\cdot(m\alpha - n\beta)'_\beta = n\sin(m\alpha - n\beta)$

$f'_\alpha\left(\frac{\pi}{2m}, 0\right) = -m\sin\frac{\pi}{2} = -m$, $f'_\beta\left(\frac{\pi}{2m}, 0\right) = n\sin\frac{\pi}{2} = n$

b)

$$z'_x = \frac{1}{x^2 - y^2}\cdot 2x$$

$$z'_y = \frac{1}{x^2 - y^2}\cdot(-2y)$$

$$z'_x(2, -1) = \frac{1}{4-1}\cdot 2$$

$$z'_y(2, -1) = \frac{1}{4-1}\cdot 2 = \frac{2}{3}$$

Naći sve parcijalne izvode prvog reda f-je

a) $z = x^2y^5 + 3x^3y - 7$

b) $z = x^y$

c) $z = (2x^2y^2 - x + 1)^3$

d) $z = \dfrac{x+y^2}{x^2+y^2+1}$

e) $z = \operatorname{arctg}\dfrac{y}{x}$

f) $u = \sqrt{x^2+y^2+z^2}$

g) $u = \ln(x^3 - y^2 + z^4)$

Rj.

a) $z'_x = 2xy^5 + 9x^2y$

$z'_y = x^2 \cdot 5y^4 + 3x^3 = 5x^2y^4 + 3x^3$

b) $z'_x = yx^{y-1}$

$z'_y = x^y \ln x$

c) $z'_x = 3(2x^2y^2 - x + 1)^2(4xy^2 - 1)$

$z'_y = 3(2x^2y^2 - x + 1)^2(4x^2y) = 12x^2y(2x^2y^2 - x + 1)^2$

d) $$z'_x = \frac{1\cdot(x^2+y^2+1) - (x+y^2)\cdot 2x}{(x^2+y^2+1)^2} = \frac{x^2+y^2+1-2x^2-2xy^2}{(x^2+y^2+1)^2} = \frac{-x^2+y^2+1-2xy^2}{(x^2+y^2+1)^2}$$

$$z'_y = \frac{2y(x^2+y^2+1) - (x+y^2)(2y)}{(x^2+y^2+1)^2} = \frac{2x^2y + \cancel{2y^3} + 2y - 2xy - \cancel{2y^3}}{(x^2+y^2+1)^2} = \frac{2y(x^2-x+1)}{(x^2+y^2+1)^2}$$

e) $z = \operatorname{arctg}\dfrac{y}{x}$

$$z'_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{y}{x}\right)'_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right) = \frac{(-1)\cdot y}{\left(1+\frac{y^2}{x^2}\right)\cdot x^2} = \frac{-y}{x^2+y^2}$$

$$z'_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x} = \frac{1}{\left(1+\frac{y^2}{x^2}\right)\cdot x} = \frac{x}{x^2+y^2}$$

f) $u = \sqrt{x^2+y^2+z^2}$, $\quad u'_x = \dfrac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2x = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$

$u'_y = \dfrac{1}{2\sqrt{x^2+y^2+z^2}}\cdot 2y = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$, $\quad u'_z = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$

g) $u = \ln(x^3-y^2+z^4)$, $\quad u'_x = \dfrac{3x^2}{x^3-y^2+z^4}$, $\quad u'_y = \dfrac{-2y}{x^3-y^2+z^4}$, $\quad u'_z = \dfrac{4z^3}{x^3-y^2+z^4}$

(#) Proveriti da li f-ja $z = x\ln\frac{y}{x}$ zadovoljava jednakost

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

Rj.

$$\frac{\partial z}{\partial x} = 1\cdot\ln\frac{y}{x} + x\cdot\frac{1}{\frac{y}{x}}\cdot\left(\frac{y}{x}\right)'_x = \ln\frac{y}{x} + \frac{x^2}{y}\cdot(-1)y(x)^{-2} = \ln\frac{y}{x} - 1$$

F-ju z možemo napisati i u obliku $z = x(\ln y - \ln x)$

$$\frac{\partial z}{\partial y} = x\cdot\frac{1}{y} = \frac{x}{y}$$

$$x\cdot\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = x\left(\ln\frac{y}{x} - 1\right) + y\cdot\frac{x}{y} = x\ln\frac{y}{x} - x + x = x\ln\frac{y}{x} = z$$

F-ja $z = x\ln\frac{y}{x}$ zadovoljava datu jednakost.

(#) Ako je $z = x^y\cdot y^x$ dokazati da je

$$x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = z\cdot(x + y + \ln z).$$

Rj.

$$\frac{\partial z}{\partial x} = yx^{y-1}\cdot y^x + x^y\cdot y^x\ln y$$

$$\frac{\partial z}{\partial y} = x^y\ln x\cdot y^x + x^y\cdot x y^{x-1}$$

$$x\cdot\frac{\partial z}{\partial x} = xy\,x^{y-1}y^x + x\ln y\,x^y y^x = y\,x^y y^x + x\ln y\,x^y y^x$$

$$y\cdot\frac{\partial z}{\partial y} = y\ln x\cdot x^y y^x + x\cdot x^y y^x$$

$$x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y} = y\,x^y y^x + \ln y^x\cdot x^y y^x + x^y y^x\ln x^y + x\,x^y y^x =$$

$$= x^y y^x\left(y + \ln(x^y\cdot y^x) + x\right) = z\cdot(x + y + \ln z)$$

što je i trebalo dobiti

Zadaci za vježbu

Naći parcijalne izvode sljedećih f-ja

1. $z = (5x^3y^3+1)^3$ 2. $r = \sqrt{ax^2 - by^2}$

3. $v = \ln(x + \sqrt{x^2+y^2})$ 4. $\rho = \arcsin \frac{x}{t}$

5. $f(m,n) = (2m)^{3n}$; izračunati f'_m i f'_n u tački $A(\frac{1}{2}; 2)$

6. $\rho(x,y,z) = \sin^2(3x+2y-z)$; izračunati $\rho'_x(1;-1;1)$, $\rho'_y(1;1;4)$, $\rho'_z(-\frac{1}{2}; 0; -1)$

7. Provjeriti da li f-ja $v = x^y$ zadovoljava jednakost

$$\frac{x}{y} \cdot \frac{\partial v}{\partial x} + \frac{1}{\ln x} \cdot \frac{\partial v}{\partial y} = 2v$$

8. Provjeriti da li f-ja $w = x + \frac{x-y}{y-z}$ zadovoljava jednakost

$$\frac{\partial w}{\partial x} + \frac{\partial w}{\partial y} + \frac{\partial w}{\partial z} = 1.$$

Rješenja:

1. $z'_x = 45x^2y^2(5x^3y^2+1)^2$;
$z'_y = 30x^3y(5x^3y^2+1)^2$.

2. $\frac{\partial r}{\partial x} = \frac{ax}{r}$; $\frac{\partial r}{\partial y} = -\frac{by}{r}$.

3. $\frac{\partial v}{\partial x} = \frac{1}{\sqrt{x^2+y^2}}$;
$\frac{\partial v}{\partial y} = \frac{y}{(x+\sqrt{x^2+y^2})\sqrt{x^2+y^2}}$.

4. $\frac{\partial \rho}{\partial x} = \frac{|t|}{t\sqrt{t^2-x^2}}$;
$\frac{\partial \rho}{\partial t} = -\frac{x}{|t|\sqrt{t^2-x^2}}$.

5. 12; 0.

6. 0; $2\sin 2$; $-\sin(-1)$

Diferenciranje f-ja više promjenjivih

Posmatrajmo f-ju tri promjenjive $u=f(x,y,z)$. Diferencijal f-je u označavamo sa du i računamo po formuli

$$du = d_x u + d_y u + d_z u = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

gdje su $d_x u$, $d_y u$, $d_z u$ parcijalni diferencijali f-je u redom po promjenjivim x, y i z.

$$d_x u = \frac{\partial u}{\partial x} dx, \quad d_y u = \frac{\partial u}{\partial y} dy, \quad d_z u = \frac{\partial u}{\partial z} dz.$$

(#) Odrediti totalne diferencijale f-ja

a) $z = 3x^2y^5$ b) $u = 2x^{yz}$ c) $p = \arccos\frac{1}{uv}$

Rj.

a) Parcijalni izvodi su

$$\frac{\partial z}{\partial x} = 6xy^5, \quad \frac{\partial z}{\partial y} = 15x^2y^4$$

Totalni diferencijal je $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$ tj.

$$dz = 6xy^5dx + 15x^2y^4dy$$

b) Parcijalni izvodi su

$$\frac{\partial u}{\partial x} = 2yzx^{yz-1}, \quad \frac{\partial u}{\partial y} = 2x^{yz}\ln x \cdot z, \quad \frac{\partial u}{\partial z} = 2yx^{yz}\ln x$$

Totalni diferencijal je

$$du = 2yzx^{yz-1}dx + 2zx^{yz}\ln x\, dy + 2yx^{yz}\ln x\, dz$$

$$= 2x^{yz}\left(\frac{yz}{x}dx + z\ln x\, dy + y\ln x\, dz\right)$$

c) Parcijalni izvodi su

$$\frac{\partial p}{\partial u} = \frac{-1}{\sqrt{1-\left(\frac{1}{uv}\right)^2}}\cdot\left(\frac{1}{uv}\right)'_u = \frac{-1}{\sqrt{\frac{u^2v^2-1}{u^2v^2}}}(-1)(uv)^{-2}\cdot v = \frac{|uv|}{u^2v\sqrt{u^2v^2-1}};$$

$$\frac{\partial p}{\partial v} = \frac{-1}{\sqrt{1-\frac{1}{u^2v^2}}}(-1)(uv)^{-2}\cdot u = \frac{u}{\sqrt{\frac{u^2v^2-1}{u^2v^2}}}\cdot\frac{1}{u^2v^2} = \frac{|uv|}{uv^2\sqrt{u^2v^2-1}};$$

Totalni diferencijal

$$dp = \frac{1}{\sqrt{u^2v^2-1}}\left(\frac{|uv|}{u^2v}du - \frac{|uv|}{uv^2}dv\right) = \frac{1}{\sqrt{u^2v^2-1}}\left(\frac{|v|\,du}{v\,|u|} - \frac{|u|\,dv}{u\,|v|}\right).$$

Odrediti parcijalne diferencijale f-je $z = \sqrt[3]{x^3 + y^3}$.

Rj.

$$z'_x = \frac{\partial z}{\partial x} = \frac{1}{3}(x^3+y^3)^{-\frac{2}{3}} \cdot 3x^2 = \frac{x^2}{\sqrt[3]{(x^3+y^3)^2}}$$

$$z'_y = \frac{\partial z}{\partial y} = \frac{1}{3}(x^3+y^3)^{-\frac{2}{3}} \cdot 3y^2 = \frac{y^2}{\sqrt[3]{(x^3+y^3)^2}}$$

dobijeni izrazi za parcijalne izvode nisu definisani u tački $(0,0)$. Izvode u toj tački treba odrediti po definiciji

$$z'_x(0,0) = \lim_{\varepsilon\to 0} \frac{z(0+\varepsilon,0) - z(0,0)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\sqrt[3]{\varepsilon^3+0^3} - 0}{\varepsilon} = \lim_{\varepsilon\to 0} 1 = 1$$

$$z'_y(0,0) = \lim_{\varepsilon\to 0} \frac{z(0,0+\varepsilon) - z(0,0)}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\sqrt[3]{0^3+\varepsilon^3} - 0}{\varepsilon} = \lim_{\varepsilon\to 0} \frac{\varepsilon}{\varepsilon} = 1$$

f-ja f ima parcijalne izvode u svim tačkama iz oblasti definisanosti. Parcijalni diferencijali su

$$d_x z = \frac{\partial z}{\partial x} dx = \begin{cases} \dfrac{x^2}{\sqrt[3]{(x^3+y^3)^2}}\, dx, & (x,y) \neq (0,0) \\ dx, & (x,y) = (0,0) \end{cases}$$

$$d_y z = \frac{\partial z}{\partial y} dy = \begin{cases} \dfrac{y^2}{\sqrt[3]{(x^3+y^3)^2}}\, dy, & (x,y) \neq (0,0) \\ dy, & (x,y) = (0,0) \end{cases}$$

Odrediti totalni diferencijal f-je $z = \arcsin\frac{x}{y}$ u tački (4,5)

Rj. F-ja je definisana za $\left|\frac{x}{y}\right| < 1$

$$\frac{\partial z}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(\frac{x}{y}\right)'_x = \frac{1}{y\sqrt{1-\frac{x^2}{y^2}}} = \frac{1}{\sqrt{y^2-x^2}}, \quad \frac{\partial z}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\cdot\left(-\frac{x}{y^2}\right) = \frac{-x}{y\sqrt{y^2-x^2}}$$

$$dz = \frac{1}{\sqrt{y^2-x^2}}dx + \frac{-x}{y\sqrt{y^2-x^2}}dy = \frac{y\,dx - x\,dy}{y\sqrt{y^2-x^2}}$$

Stavljajući u dobijeni izraz $x=4$ i $y=5$ dobijemo $dz = \frac{1}{15}(5dx - 4dy)$

Pomoću totalnog diferencijala približno izračunati $\ln(\sqrt[3]{1{,}03} + \sqrt[4]{0{,}98} - 1)$.

Rj. Neka je $z = \ln(\sqrt[3]{x} + \sqrt[4]{y} - 1)$ gdje je $x = a+\epsilon = 1+0{,}03$ i $y = b+\omega = 1-0{,}02$.

Tada je $z(a,b) = \ln(\sqrt{1} - \sqrt{1} - 1) = \ln 1 = 0$; $z = z(a,b) + \Delta z$.

($\Delta z = f(a+\epsilon, b+\omega) - f(a,b)$ totalni priraštaj f-je u tački (a,b)).

Kako je $\Delta z \approx dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{1}{\sqrt[3]{x}+\sqrt[4]{y}-1}\left(\frac{1}{3\sqrt[3]{x^2}}dx + \frac{1}{4\sqrt[4]{y^3}}dy\right) =$

$= \frac{1}{1}\left(\frac{1}{3}\cdot 0{,}03 - \frac{1}{4}\cdot 0{,}02\right) = 0{,}005$. Pa $z = z_0 + \Delta z \approx 0{,}0005$.

Naći totalni diferencijal i totalni priraštaj f-je $z = x^2+y^2+xy$ pri prelazu od tačke (1,1) u tačku (1,1; 0,9).

Rj. po definiciji totalnog priraštaja dobijemo

$$\Delta z = f(x+\Delta x, y+\Delta y) - f(x,y) = (x+\Delta x)^2 + (y+\Delta y)^2 + (x+\Delta x)(y+\Delta y) - (x^2+y^2+xy) =$$

$$= x^2 + 2x\Delta x + \Delta x^2 + y^2 + 2y\Delta y + \Delta y^2 + xy + x\Delta y + y\Delta x + \Delta x\Delta y - x^2 - y^2 - xy$$

$$= 2x\Delta x + \Delta x^2 + y\Delta x + 2y\Delta y + \Delta y^2 + x\Delta y + \Delta x\Delta y = (2x+y+\Delta x)\Delta x + (2y+x+\Delta x+\Delta y)\Delta y$$

Ako ubacimo u formulu vrijednosti $x=1$, $y=1$, $\Delta x = 1{,}1-1 = 0{,}1$, $\Delta y = 0{,}9-1 = -0{,}1$ dobićemo totalni priraštaj date f-je u tački (1,1)

$$\Delta z = (2+1+0{,}1)\,0{,}1 + (2+1+0{,}1-0{,}1)\cdot(-0{,}1) = 3{,}1\cdot 0{,}1 + 3\cdot(-0{,}1) = 0{,}31 - 0{,}3 = 0{,}01$$

$$dz = (2x+y)dx + (2y+x)dy \qquad dz = (2+1)\cdot 0{,}1 + (2+1)\cdot(-0{,}1) = 0{,}3 - 0{,}3 = 0$$

Diferenciranje složenih f-ja

F-ju z nazivamo složenom f-jom od tri nezavisno promjenjive x, y, t ako je ona zadana putem argumenata $u, v, \ldots, w$:

$$z = F(u, v, \ldots, w)$$

gdje je

$$u = f(x, y, t), \quad v = \varphi(x, y, t), \quad \ldots, \quad w = \psi(x, y, t).$$

Slično bi definisali f-ju od n nezavisno promjenjivih.

Parcijalni izvod složene f-je po jednoj od nezavisnih promjenjivih jednak je sumi proizvoda parcijalnog izvoda f-je po njenom argumentu sa parcijalnim izvodom istog argumenta po nezavisnoj promjenjivoj:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} + \ldots + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial x};$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} + \ldots + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial y}; \qquad \ldots(*)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial t} + \ldots + \frac{\partial z}{\partial w} \cdot \frac{\partial w}{\partial t}.$$

Ako su svi argumenti $u, v, \ldots, w$ f-je jedne nazavisno promjenjive x, tada je i z složena f-ja po promjenjivoj x. Izvod takve složene f-je (od jedne nazavisno promjenjive) naziva se <u>totalni izvod</u> i dat je preko formule

$$\frac{dz}{dx} = \frac{\partial z}{\partial u} \cdot \frac{du}{dx} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dx} + \ldots + \frac{\partial z}{\partial w} \cdot \frac{dw}{dx}. \qquad \ldots(**)$$

(dobije se iz formule totalnog diferencijala f-je $z(u, v, w)$ tako što je podjelimo sa dx).

(#) Naći izvode složenih f-ja

a) $y=u^2e^v$, $u=\sin x$, $v=\cos x$;

b) $p=u^v$, $u=\ln(x-y)$, $v=e^{\frac{x}{y}}$;

c) $z=x\sin v\cos w$, $v=\ln(x^2+1)$, $w=-\sqrt{1-x^2}$.

Rj: a) Primjetimo da je y složena f-ja po nezavisno promjenjivoj x. Koristimo formulu (**)

$$\frac{dy}{dx}=\frac{\partial y}{\partial u}\cdot\frac{du}{dx}+\frac{\partial y}{\partial v}\cdot\frac{dv}{dx}\overset{(\square)}{=}2ue^v\cos x-u^2e^v\sin x$$

$$\frac{\partial y}{\partial u}=2ue^v,\quad \frac{du}{dx}=\cos x,\quad \frac{\partial y}{\partial v}=u^2e^v,\quad \frac{dv}{dx}=-\sin x \qquad \ldots(\square)$$

b) p je složena f-ja dvije promjenjive x, y. Koristimo formulu (*)

$$\frac{\partial p}{\partial x}=\frac{\partial p}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial p}{\partial v}\cdot\frac{\partial v}{\partial x}=vu^{v-1}\cdot\frac{1}{x-y}+u^v\ln u\cdot\frac{1}{y}e^{\frac{x}{y}}$$

$$\frac{\partial p}{\partial y}=\frac{\partial p}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial p}{\partial v}\cdot\frac{\partial v}{\partial y}=vu^{v-1}\cdot\frac{1}{y-x}+u^v\ln u\left(-\frac{x}{y^2}e^{\frac{x}{y}}\right)$$

c) z je složena f-ja jedne promjenjive x. Koristimo formulu (**).

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}+\frac{\partial z}{\partial w}\cdot\frac{dw}{dx}$$

$$\frac{dz}{dx}=\sin v\cos w+x\cos v\cos w\cdot\frac{2x}{x^2+1}-x\sin v\sin w\,\frac{x}{\sqrt{1-x^2}}$$

Nađi diferencijal f-je u (nađi du) ako je $u=f(\sqrt{x^2+y^2})$.

Rj. $u=f(\sqrt{x^2+y^2})$, uvedimo oznaku $t=\sqrt{x^2+y^2}$.

$$u=f(t)=f(t(x,y)),\qquad du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy$$

$$\frac{\partial u}{\partial x}=f'_t\cdot\frac{\partial t}{\partial x}=f'_t\,\frac{2x}{2\sqrt{x^2+y^2}}=\frac{x f'_t}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y}=f'_t\cdot\frac{\partial t}{\partial y}=f'_t\cdot\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y\cdot f'_t}{\sqrt{x^2+y^2}}$$

$$du=\frac{f'_{\sqrt{x^2+y^2}}(\sqrt{x^2+y^2})(x\,dx+y\,dy)}{\sqrt{x^2+y^2}}$$

Ako je $z=\dfrac{y}{f(x^2-y^2)}$ tada je $\dfrac{1}{x}\cdot\dfrac{\partial z}{\partial x}+\dfrac{1}{y}\cdot\dfrac{\partial z}{\partial y}=\dfrac{z}{y^2}$. Dokazati.

Rj.

$$z=\frac{y}{f(\xi)}\quad \text{gdje je } \xi=x^2-y^2$$

$$\frac{\partial z}{\partial x}=\frac{0\cdot f(\xi)-y\cdot\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial x}}{f^2(\xi)}=\frac{-2xy\,\frac{\partial f}{\partial \xi}}{f^2(\xi)}$$

$$\frac{\partial z}{\partial y}=\frac{1\cdot f(\xi)-y\cdot\frac{\partial f}{\partial \xi}\cdot\frac{\partial \xi}{\partial y}}{f^2(\xi)}=\frac{f(\xi)+2y^2\frac{\partial f}{\partial \xi}}{f^2(\xi)}$$

(#) Ako je $x^2 = v\cdot w$, $y^2 = u\cdot w$, $z^2 = u\cdot v$ i $f(x,y,z) = F(u,v,w)$

dokazati $x\cdot\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} + z\cdot\frac{\partial f}{\partial z} = u\cdot\frac{\partial F}{\partial u} + v\cdot\frac{\partial F}{\partial v} + w\cdot\frac{\partial F}{\partial w}$.

Rj: $F(u,v,w) = f(x,y,z) = f(\sqrt{v\cdot w}, \sqrt{u\cdot w}, \sqrt{u\cdot v})$

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial u} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial u}$$

$$= f'_x\cdot 0 + f'_y\cdot\frac{\sqrt{w}}{2\sqrt{u}} + f'_z\cdot\frac{\sqrt{v}}{2\sqrt{u}} = f'_y\cdot\frac{\sqrt{w}}{2\sqrt{u}} + f'_z\cdot\frac{\sqrt{v}}{2\sqrt{u}}$$

$$\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\sqrt{w}}{2\sqrt{v}} + \frac{\partial f}{\partial y}\cdot 0 + \frac{\partial f}{\partial z}\cdot\frac{\sqrt{u}}{2\sqrt{v}} = f'_x\cdot\frac{\sqrt{w}}{2\sqrt{v}} + f'_z\cdot\frac{\sqrt{u}}{2\sqrt{v}}$$

$$\frac{\partial F}{\partial w} = \frac{\partial f}{\partial x}\cdot\frac{\sqrt{v}}{2\sqrt{w}} + \frac{\partial f}{\partial y}\cdot\frac{\sqrt{u}}{2\sqrt{w}} + \frac{\partial f}{\partial z}\cdot 0 = f'_x\cdot\frac{\sqrt{v}}{2\sqrt{w}} + f'_y\cdot\frac{\sqrt{u}}{2\sqrt{w}}$$

$$u\cdot\frac{\partial F}{\partial u} = \frac{\partial f}{\partial y}\cdot\frac{\sqrt{uw}}{2} + \frac{\partial f}{\partial z}\cdot\frac{\sqrt{uv}}{2}$$

$$v\cdot\frac{\partial F}{\partial v} = \frac{\partial f}{\partial x}\cdot\frac{\sqrt{vw}}{2} + \frac{\partial f}{\partial z}\cdot\frac{\sqrt{uv}}{2}$$

$$w\cdot\frac{\partial F}{\partial w} = \frac{\partial f}{\partial x}\cdot\frac{\sqrt{vw}}{2} + \frac{\partial f}{\partial y}\cdot\frac{\sqrt{uw}}{2}$$

Prema tome $x\cdot\frac{\partial f}{\partial x} + y\cdot\frac{\partial f}{\partial y} + z\cdot\frac{\partial f}{\partial z} = u\frac{\partial F}{\partial u} + v\frac{\partial F}{\partial v} + w\cdot\frac{\partial F}{\partial w}$

q.e.d.

(#) ISPITNI ZADATAK

Ako je $z=z(x,y)$ i $x+y+z=f(x^2+y^2+z^2)$ provjeriti da li je tačna jednakost

$$(y-z)\cdot\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=x-y.$$

Rj. $z=z(x,y) \Rightarrow z$ je f-ja dvije promjenjive x i y.

$$z=f(x^2+y^2+z^2)-x-y$$

$$t=x^2+y^2+z^2$$

$$s=-x-y$$

$$z=f(t)+s$$

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial x}+\frac{\partial s}{\partial x}$$

$$\frac{\partial z}{\partial x}=f'_t\cdot\left(2x+2z\frac{\partial z}{\partial x}\right)-1$$

$$\frac{\partial z}{\partial x}-f'_t\cdot 2z\frac{\partial z}{\partial x}=f'_t\cdot 2x-1$$

$$\frac{\partial z}{\partial x}=\frac{2xf'_t-1}{1-2zf'_t}$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial y}+\frac{\partial s}{\partial y}$$

$$\frac{\partial z}{\partial y}=f'_t\cdot\left(2y+2z\frac{\partial z}{\partial x}\right)-1$$

$$\frac{\partial z}{\partial y}-2zf'_t\frac{\partial z}{\partial x}=2yf'_t-1$$

$$\frac{\partial z}{\partial y}=\frac{2yf'_t-1}{1-2zf'_t}$$

$$(y-z)\cdot\frac{\partial z}{\partial x}+(z-x)\frac{\partial z}{\partial y}=\frac{(y-z)(2xf'_t-1)}{1-2zf'_t}+\frac{(z-x)(2yf'_t-1)}{1-2zf'_t}=$$

$$=\frac{\cancel{2xyf'_t}-y-2xzf'_t+z+2yzf'_t-z\cancel{-2xyf'_t}+x}{1-2zf'_t}=$$

$$=\frac{(x-y)-2xzf'_t+2yzf'_t}{1-2zf'_t}=\frac{(x-y)+2zf'_t(-x+y)}{1-2zf'_t}=$$

$$=\frac{(x-y)(1-2zf'_t)}{1-2zf'_t}=x-y$$

(#) Ako je $z = \frac{y}{f(x^2-y^2)}$, gdje je f diferencijabilna f-ja,

izračunati $\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y}$.

Rj. $z = y f^{-1}(x^2-y^2) = y f^{-1}(u)$, gdje je $u = x^2-y^2$

$$\frac{\partial z}{\partial x} = y(-1) f_u^{-2}(x^2-y^2)\cdot 2x = \frac{-2xy}{f_u^2(x^2+y^2)}$$

$$\frac{\partial z}{\partial y} = \left(y f^{-1}(u)\right)'_y = 1\cdot f^{-1}(u) + y\cdot(-1) f_u^{-2}(u)\cdot(-2y) =$$

$$= \frac{1}{f(x^2-y^2)} + \frac{2y^2}{f_u^2(x^2+y^2)}$$

$$\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = \frac{-2y}{f_u^2(x^2+y^2)} + \frac{1}{y f(x^2-y^2)} + \frac{2y}{f_u^2(x^2+y^2)} =$$

$$= \frac{1}{y f(x^2-y^2)} = \frac{1}{y^2}\cdot\frac{y}{f(x^2-y^2)} = \frac{z}{y^2}$$

prema tome $\frac{1}{x}\cdot\frac{\partial z}{\partial x} + \frac{1}{y}\cdot\frac{\partial z}{\partial y} = \frac{z}{y^2}$.

(#) Ako je $z = e^{y}\varphi\left(y e^{\frac{x^2}{2y^2}}\right)$ gdje je φ diferencijabilna f-ja, dokazati da je $(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = xyz$.

Rj.

$$z = e^{y}\varphi(\xi), \text{ gdje je } \xi(x,y) = y e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial \xi}{\partial x} = y e^{\frac{x^2}{2y^2}} \cdot 2 \cdot \frac{x}{2y^2} = \frac{x}{y} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial \xi}{\partial y} = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}}\left(\frac{1}{2}x^2y^{-2}\right)'_y = e^{\frac{x^2}{2y^2}} + y e^{\frac{x^2}{2y^2}}\left(\frac{1}{2}x^2\cdot(-2)y^{-3}\right)$$

$$= e^{\frac{x^2}{2y^2}} - \frac{x^2}{y^2} e^{\frac{x^2}{2y^2}}$$

$$\frac{\partial z}{\partial x} = e^{y}\cdot\frac{\partial \varphi}{\partial \xi}\cdot\frac{\partial \varphi}{\partial x} = \frac{x}{y} e^{y} e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$\frac{\partial z}{\partial y} = e^{y}\varphi(\xi) + e^{y}\cdot\frac{\partial \varphi}{\partial \xi}\cdot\frac{\partial \xi}{\partial y} = e^{y}\varphi(\xi) + e^{y}e^{\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi} -$$
$$- e^{y}\cdot\frac{x^2}{y^2} e^{\frac{x^2}{2y^2}} \frac{\partial \varphi}{\partial \xi}$$

$$(x^2-y^2)\frac{\partial z}{\partial x} + xy\frac{\partial z}{\partial y} = (x^2-y^2)\cdot\frac{x}{y} e^{y+\frac{x^2}{2y^2}}\frac{\partial \varphi}{\partial \xi} +$$
$$+ xy\left(e^{y}\varphi(\xi) + e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi} - \frac{x^2}{y^2} e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi}\right)$$
$$= \underline{\frac{x^3}{y} e^{y+\frac{x^2}{2y^2}}\frac{\partial \varphi}{\partial \xi}} \underline{\underline{- yx e^{y+\frac{x^2}{2y^2}}\frac{\partial \varphi}{\partial \xi}}} + xy e^{y}\varphi(\xi) +$$
$$\underline{\underline{+ xy e^{y+\frac{x^2}{2y^2}}\frac{\partial \varphi}{\partial \xi}}} \underline{- \frac{x^3}{y} e^{y+\frac{x^2}{2y^2}}\cdot\frac{\partial \varphi}{\partial \xi}} =$$

$$= xy e^{y}\varphi(\xi) = xy e^{y}\varphi\left(y e^{\frac{x^2}{2y^2}}\right) = xyz$$

Parcijalni izvodi i diferencijali višeg reda f-ja dvije i više promjenjivih

Parcijalnim izvodima drugog reda f-je $z = f(x,y)$ nazivamo parcijalnim izvodima njenih parcijalnih izvoda prvog reda. Za parcijalne izvode drugog reda upotrebljavamo ove oznake

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = f''_{xx}(x,y)$$

∂ DELTA

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x \partial y} = f''_{xy}(x,y) \quad \text{itd.}$$

Analogno se definiraju i označavaju izvodi viših redova.

Diferencijalom drugog reda f-je $z = f(x,y)$ nazivamo diferencijal diferencijala prvog reda te f-je za fiksirane priraste nezavisnih varijabli.

$$d^2 z = d(dz)$$

Analogno se određuju diferencijali f-je z višega nego drugog reda, na primjer $d^3 z = d(d^2 z)$

i općenito $d^n z = d(d^{n-1} z)$ $(n = 2, 3, \dots)$

Ako je $z = f(x,y)$ gdje su x i y nezavisne varijable i f-ja f ima neprekidne parcijalne izvode drugog reda, tada se diferencijal drugog reda f-je z računa po formuli

$$d^2 z = \frac{\partial^2 z}{\partial x^2} dx^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx dy + \frac{\partial^2 z}{\partial y^2} dy^2.$$

Općenito, kada postoje neprekidne odgovarajuće derivacije, vrijedi simbolička formula

$$d^n z = \left(dx\frac{\partial}{\partial x} + dy\frac{\partial}{\partial y}\right)^n z,$$

koja se formalno razvije po binomnom zakonu.

Naći parcijalne izvode drugog reda f-je

a) $z=e^{-xy}$ c) $u=x^3y+y^3x+z^3y$ e)$^\vee$ $z=\ln tg\frac{x}{y}$

b) $z=x^3+y^3-xy$ d)$^\vee$ $u=\ln(x+y-z)$ f)$^\vee$ $u=\sin(x^2+y+z^3)$

Rj: a) $z=e^{-xy}$

$$\frac{\partial z}{\partial x}=e^{-xy}\cdot(-y)=-ye^{-xy}$$

$$\frac{\partial z}{\partial y}=e^{-xy}\cdot(-x)=-xe^{-xy}$$

$$\frac{\partial^2 z}{\partial x^2}=(-y)e^{-xy}\cdot(-y)=y^2e^{-xy}$$

$$\frac{\partial^2 z}{\partial y^2}=(-x)e^{-xy}\cdot(-x)=x^2e^{-xy}$$

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}=-e^{-xy}-ye^{-xy}(-x)=$$

$$=e^{-xy}(xy-1)$$

b) $z=x^3+y^3-xy$

$$\frac{\partial z}{\partial x}=3x^2-y$$

$$\frac{\partial z}{\partial y}=3y^2-x$$

$$\frac{\partial^2 z}{\partial x^2}=6x,\quad \frac{\partial^2 z}{\partial y^2}=6y,\quad \frac{\partial^2 z}{\partial x\partial y}=-1$$

c) $u=x^3y+y^3x+z^2y$

$$\frac{\partial u}{\partial x}=3x^2y+y^3$$

$$\frac{\partial u}{\partial y}=x^3+3y^2x+z^3$$

$$\frac{\partial u}{\partial z}=3z^2y$$

$$\frac{\partial^2 u}{\partial x^2}=6xy,\quad \frac{\partial^2 u}{\partial y^2}=6xy,\quad \frac{\partial^2 u}{\partial z^2}=6yz$$

$$\frac{\partial^2 u}{\partial x\partial y}=3x^2+3y^2,\quad \frac{\partial^2 u}{\partial x\partial z}=0$$

$$\frac{\partial^2 u}{\partial y\partial z}=3z^2$$

d) $u=\ln(x+y-z)$

$$\frac{\partial u}{\partial x}=\frac{1}{x+y-z}$$

$$\frac{\partial u}{\partial y}=\frac{1}{x+y-z}$$

$$\frac{\partial u}{\partial z}=\frac{-1}{x+y-z}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{-1}{(x+y-z)^2}$$

$$\frac{\partial^2 u}{\partial y^2}=\frac{-1}{(x+y-z)^2}$$

$$\frac{\partial^2 u}{\partial z^2}=\frac{-1}{(x+y-z)^2}$$

završiti sami

...

Provjeriti da li vrijedi

a) $u=\ln(x^2+y^2) \implies \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0$

b) $u=e^{-\alpha x}\cdot\varphi(x-y) \implies \dfrac{\partial^2 u}{\partial x^2}-\dfrac{\partial^2 u}{\partial y^2}-2\alpha\cdot\dfrac{\partial u}{\partial y}=\alpha^2 u$

Rj. a) $\dfrac{\partial u}{\partial x}=\dfrac{2x}{x^2+y^2}$ $\qquad \dfrac{\partial^2 u}{\partial x^2}=\dfrac{2(x^2+y^2)-2x\cdot(2x)}{(x^2+y^2)^2}=\dfrac{2y^2-2x^2}{(x^2+y^2)^2}$

$\dfrac{\partial u}{\partial y}=\dfrac{2y}{x^2+y^2}$ $\qquad \dfrac{\partial^2 u}{\partial y^2}=\dfrac{2(x^2+y^2)-2y(2y)}{(x^2+y^2)^2}=\dfrac{2x^2-2y^2}{(x^2+y^2)^2}$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{2y^2-2x^2}{(x^2+y^2)^2}+\frac{2x^2-2y^2}{(x^2+y^2)^2}=\frac{2y^2-2x^2+2x^2-2y^2}{(x^2+y^2)^2}=0$$ što je i trebalo dobiti

b) $u=e^{-\alpha x}\cdot\varphi(x-y)$

$$\frac{\partial u}{\partial x}=e^{-\alpha x}\cdot(-\alpha)\,\varphi(x-y)+e^{-\alpha x}\cdot\varphi'_x=e^{-\alpha x}\left[-\alpha\varphi(x-y)+\varphi'_x\right]$$

$$\frac{\partial^2 u}{\partial x^2}=e^{-\alpha x}\cdot(-\alpha)\left(-\alpha\varphi(x-y)+\varphi'_x\right)+e^{-\alpha x}\left[-\alpha\varphi'_x+\varphi''_{xx}\right]$$
$$=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-\alpha\varphi'_x-\alpha\varphi'_x+\varphi''_{xx}\right)=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-2\alpha\varphi'_x+\varphi''_{xx}\right)$$

$$\frac{\partial u}{\partial y}=e^{-\alpha x}\cdot\varphi'_y\cdot(-1)=-e^{\alpha x}\varphi'_y$$

$$\frac{\partial^2 u}{\partial y^2}=-e^{\alpha x}\varphi''_{yy}\cdot(-1)=e^{\alpha x}\cdot\varphi''_{yy}$$

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial y^2}-2\alpha\frac{\partial u}{\partial y}=e^{-\alpha x}\left(\alpha^2\varphi(x-y)-2\alpha\varphi'_x+\varphi''_{xx}-\varphi''_{yy}+2\alpha\varphi'_y\right)=$$ (u slučaju da je $\varphi'_x=\varphi'_y$ i $\varphi''_{xx}=\varphi''_{yy}$) $=\alpha^2e^{-\alpha x}\varphi(x-y)=\alpha^2 u$

#) Naći parcijalne izvode prvog i drugog reda f-je $z=\ln(x^2+y^2)$.

Rj.

$$\frac{\partial z}{\partial x} = \frac{1}{x^2+y^2}\left(x^2+y^2\right)'_x = \frac{2x}{x^2+y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2+y^2}\left(x^2+y^2\right)'_y = \frac{2y}{x^2+y^2}$$

$$\frac{\partial^2 z}{\partial x^2} = \left(\frac{2x}{x^2+y^2}\right)'_x = 2\left(\frac{x}{x^2+y^2}\right)'_x = 2\,\frac{1\cdot(x^2+y^2)-x\cdot 2x}{(x^2+y^2)^2}$$

$$= 2\cdot\frac{y^2-x^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial x\partial y} = \left(\frac{2x}{x^2+y^2}\right)'_y = 2\,\frac{1\cdot(x^2+y^2)-x\cdot 2y}{(x^2+y^2)^2} = 2\cdot\frac{x^2-2xy+y^2}{(x^2+y^2)^2}$$

$$= 2\cdot\frac{(x-y)^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \left(\frac{2y}{x^2+y^2}\right)'_y = 2\left(\frac{y}{x^2+y^2}\right)'_y = 2\,\frac{1\cdot(x^2+y^2)-y\cdot 2y}{(x^2+y^2)^2}$$

$$= 2\,\frac{x^2-y^2}{(x^2+y^2)^2}$$

Parcijalni izvodi višeg reda složenih f-ja

Ako je $u = \varphi(\xi, \eta)$ pričemu je $\xi = x + y$, $\eta = x - y$ izračunati izvode $\frac{\partial^2 u}{\partial x^2}$, $\frac{\partial^2 u}{\partial x \partial y}$, $\frac{\partial^2 u}{\partial y^2}$.

Rj.:

$$\frac{\partial u}{\partial x} = \frac{\partial \varphi}{\partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \varphi}{\partial \eta} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial \varphi}{\partial \xi} + \frac{\partial \varphi}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial \xi}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial \eta}\right) = \frac{\partial^2 \varphi}{\partial \xi^2} \frac{\partial \xi}{\partial x} + \frac{\partial^2 \varphi}{\partial \xi \partial \eta} \frac{\partial \eta}{\partial x} +$$

$$+ \frac{\partial^2 \varphi}{\partial \eta \partial \xi} \cdot \frac{\partial \xi}{\partial x} + \frac{\partial^2 \varphi}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial x} = \frac{\partial^2 \varphi}{\partial \xi^2} + 2 \frac{\partial^2 \varphi}{\partial \xi \partial \eta} + \frac{\partial^2 \varphi}{\partial \eta^2}$$

$$= \left(\frac{\partial \varphi}{\partial \xi} + \frac{\partial \varphi}{\partial \eta}\right)^2$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial \xi}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial \eta}\right) = \frac{\partial^2 \varphi}{\partial \xi^2} \frac{\partial \xi}{\partial y} + \frac{\partial^2 \varphi}{\partial \xi \partial \eta} \cdot \frac{\partial \eta}{\partial y} +$$

$$+ \frac{\partial^2 \varphi}{\partial \eta \partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial^2 \varphi}{\partial \eta^2} \cdot \frac{\partial \eta}{\partial y} = \frac{\partial^2 \varphi}{\partial \xi^2} - \frac{\partial^2 \varphi}{\partial \eta^2}$$

(#) Ako je $u = \frac{\varphi(x-y) + \psi(x+y)}{x}$, gdje su φ i ψ diferencijabilne f-je izračunati $\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) - x^2 \frac{\partial^2 u}{\partial y^2}$.

Rj.

$$u = \frac{1}{x}(\varphi(x-y) + \psi(x+y)) = x^{-1}(\varphi(x-y) + \psi(x+y))$$

$$u'_x = \frac{\partial u}{\partial x} = (-1)x^{-2}(\varphi(\underbrace{x-y}_{s}) + \psi(\underbrace{x+y}_{t})) + \frac{1}{x}(\varphi'_s \cdot s'_x + \psi'_t \cdot t'_x) =$$

$$= \frac{-1}{x^2}[\varphi(x-y) + \psi(x+y)] + \frac{1}{x}(\varphi'_s \cdot 1 + \psi'_t \cdot 1)$$

$$x^2 \frac{\partial u}{\partial x} = -\varphi(x-y) - \psi(x+y) + x(\varphi'_s + \psi'_t)$$ gdje su $s = x-y$ i $t = x+y$

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) = \underline{-\varphi'_s \cdot 1} \underline{\underline{- \psi'_t \cdot 1}} \underline{+ 1 \cdot (\varphi'_s} \underline{\underline{+ \psi'_t}}) + x(\varphi''_{ss} \cdot 1 + \psi''_{tt} \cdot 1)$$

$$= x(\varphi''_{ss} + \psi''_{tt}) \quad \dots (1)$$

$$\frac{\partial u}{\partial y} = \frac{1}{x}(\varphi'_s \cdot s'_y + \psi'_t \cdot t'_y) = \frac{1}{x}(\varphi'_s \cdot (-1) + \psi'_t \cdot 1) = \frac{1}{x}(-\varphi'_s + \psi'_t)$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{x}(-\varphi''_{ss} \cdot s'_y + \psi''_{tt} \cdot t'_y) = \frac{1}{x}(\varphi''_{ss} + \psi''_{tt})$$

$$x^2 \frac{\partial^2 u}{\partial y^2} = x(\varphi''_{ss} + \psi''_{tt}) \quad \dots (2)$$

$$\frac{\partial}{\partial x}\left(x^2 \frac{\partial u}{\partial x}\right) - x^2 \frac{\partial^2 u}{\partial y^2} \overset{(1) i (2)}{=\!=} 0$$ traženo rješenje

Zadaci za vježbu

§ 3. Izvodi i diferencijali funkcija više promenljivih

Parcijalni izvodi

3032. Zapremina gasa v je funkcija njegove temperature i pritiska: $v=f(p, T)$. Kad pritisak gasa ostaje konstantan, srednjim koeficijentom širenja gasa pri promeni njegove temperature od T_1 do T_2 naziva se veličina $\frac{v_2-v_1}{v(T_2-T_1)}$. Šta treba zvati koeficijentom širenja gasa pri konstantnom pritisku za datu temperaturu T_0?

3033. Temperatura θ u datoj tački A štapa Ox je funkcija apscise x tačke A i vremena t: $\theta=f(x, t)$. Kakav fizički smisao imaju parcijalni izvodi $\frac{\partial\theta}{\partial t}$ i $\frac{\partial\theta}{\partial x}$?

3034. Površina S pravougaonika čija je osnovica b i visina h izražava se obrascem $S=bh$. Naći $\frac{\partial S}{\partial h}$ i $\frac{\partial S}{\partial x}$ i objasniti geometrijski smisao rezultata.

3035. Date su dve funkcije: $u=\sqrt{a^2-x^2}$ (a je konstanta) i $z=\sqrt{y^2-x^2}$. Naći $\frac{du}{dx}$ i $\frac{\partial z}{\partial x}$ i uporediti rezultate.

U zadacima 3036—3084 naći parcijalne izvode datih funkcija po svakoj od nezavisno promenljivih (x, y, z, u, v, t, φ i ψ su promenljive veličine).

3036. $z=x-y$.

3037. $z=x^3y-y^3x$.

3038. $\theta=axe^{-t}+bt$ (a, b su konstante).

3039. $z=\frac{u}{v}+\frac{v}{u}$.

3040. $z=\frac{x^3+y^3}{x^2+y^2}$.

3041. $z=(5x^2y-y^3+7)^3$.

3042. $z=x\sqrt{y}+\frac{y}{\sqrt[3]{x}}$.

3043. $z=\ln(x+\sqrt{x^2+y^2})$.

3044. $z=\operatorname{arctg}\frac{x}{y}$.

3045. $z=\frac{1}{\operatorname{arctg}\frac{y}{x}}$.

3046. $z=x^y$.

3047. $z=\ln(x^2+y^2)$.

3048. $z=\ln\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}$.

3049. $z=\arcsin\frac{\sqrt{x^2-y^2}}{\sqrt{x^2+y^2}}$.

3050. $z=\ln\operatorname{tg}\frac{x}{y}$.

3051. $z=e^{-\frac{x}{y}}$.

3052. $z=\ln(x+\ln y)$.

3053. $u=\operatorname{arctg}\frac{v+w}{v-w}$.

3054. $z=\sin\frac{x}{y}\cos\frac{y}{x}$.

3055. $z=\left(\frac{1}{3}\right)^{\frac{y}{x}}$.

3056. $z=(1+xy)^y$.

3057. $z=xy\ln(x+y)$.

3058. $z=x^{xy}$.

3059. $u=xyz$.

3060. $u=xy+yz+zx$.

3061. $u=\sqrt{x^2+y^2+z^2}$.

3062. $u=x^3+yz^2+3yx-x+z$.

3063. $w=xyz+yzv+zvk+vxy$.

3064. $u=e^{x(x^2+y^2+z^2)}$.

3066. $u=\ln(x+y+z)$

3065. $u=\sin(x^2+y^2+z^2)$.

3075. $z=\operatorname{arctg}\sqrt{x^y}$.

3067. $u=x^{\frac{y}{z}}$.

3068. $u=x^{yz}$.

3069. $f(x, y)=x+y-\sqrt{x^2+y^2}$ u tački $(3, 4)$.

3070. $z=\ln\left(x+\dfrac{y}{2x}\right)$ u tački $(1, 2)$.

3071. $z=(2x+y)^{2x+y}$.

3072. $z=(1+\log_y x)^3$.

3073. $z=xye^{\sin \pi xy}$.

3074. $z=(x^2+y^2)\dfrac{1-\sqrt{x^2+y^2}}{1+\sqrt{x^2-y^2}}$.

3076. $z=2\sqrt{\dfrac{1-\sqrt{xy}}{1+\sqrt{xy}}}$.

3077. $z=\ln[xy^2+yx^2+\sqrt{1+(xy^2+yx^2)^2}]$.

3078. $z=\sqrt{1-\left(\dfrac{x+y}{xy}\right)^2}+\arcsin\dfrac{x+y}{xy}$.

3079. $z=\operatorname{arctg}\left(\operatorname{arctg}\dfrac{y}{x}\right)-\dfrac{1}{2}\dfrac{\operatorname{arctg}\dfrac{x}{y}-1}{\operatorname{arctg}\dfrac{x}{y}+1}-\operatorname{arctg}\dfrac{x}{y}$.

3080. $u=\dfrac{k}{(x^2+y^2+z^2)^2}$.

3081. $u=\operatorname{arctg}(x-y)^z$.

3082. $u=(\sin x)^{yz}$.

3083. $u=\ln\dfrac{1-\sqrt{x^2+y^2+z^2}}{1+\sqrt{x^2+y^2+z^2}}$.

3084. $w=\dfrac{1}{2}\operatorname{tg}^2(x^2y^2+z^2v^2-xyzv)+\ln\cos(x^2y^2+z^2v^2-xyzv)$.

3085. $n=\dfrac{\cos(\varphi-2\psi)}{\operatorname{sos}(\varphi+2\psi)}$. Naći $\left(\dfrac{\partial u}{\partial \psi}\right)_{\substack{\varphi=\frac{\pi}{4}\\ \psi=\pi}}$

3086. $u=\sqrt{az^3-bt^3}$. Naći $\dfrac{\partial u}{\partial z}$ i $\dfrac{\partial u}{\partial t}$ za $z=b,\ t=a$.

3087. $z=\dfrac{x\cos y-y\cos x}{1+\sin x+\sin y}$. Naći $\dfrac{\partial z}{\partial x}$ i $\dfrac{\partial z}{\partial y}$ za $x=y=0$.

3088. $u=\sqrt{\sin^2 x+\sin^2 y+\sin^2 z}$. Naći $\left(\dfrac{\partial u}{\partial z}\right)_{\substack{x=0\\ y=0\\ z=\frac{\pi}{4}}}$

3089. $u=\ln(1+x+y^2+z^3)$. Naći $u'_x+u'_y+u'_z$ za $x=y=z=1$.

3090. $f(x, y)=x^3y-y^3x$. Naći $\left(\dfrac{\dfrac{\partial f}{\partial x}+\dfrac{\partial f}{\partial y}}{\dfrac{\partial f}{\partial x}\cdot\dfrac{\partial f}{\partial y}}\right)_{\substack{x=1\\ y=2}}$

3091. Koliki ugao zaklapa tangenta u tački (2, 4, 5) krive $\begin{cases} z=\dfrac{x^2+y^2}{4} \\ y=4 \end{cases}$ sa pozitivnim pravcem apscisne ose.

3092. Koliki ugao zaklapa tangenta krive $\begin{cases} z=\sqrt{1+x^2+y^2} \\ x=1 \end{cases}$ u tački $(1, 1, \sqrt{3})$ sa pozitivnim pravcem ordinatne ose.

3093. Pod kojim se uglom seku ravne krive po kojima ravan $y=2$ preseca površine $z=x^2+\dfrac{y^2}{6}$ i $z=\dfrac{x^2+y^2}{3}$?

Diferencijali. Približna računanja

U zadacima 3094—3097 naći parcijalne diferencijale datih funkcija po svakoj od nezavisno promenljivih.

3094. $z=xy^3-3x^2y^2+2y^4$.

3095. $z=\sqrt{x^2+y^2}$.

3096. $z=\dfrac{xy}{x^2+y^2}$.

3097. $u=\ln(x^3+2y^3-z^3)$.

3098. $z=\sqrt[3]{x+y^2}$. Naći $d_y z$ za $x=2$, $y=5$, $\Delta y=0{,}01$.

3099. $z=\sqrt{\ln xy}$. Naći $d_x z$ za $x=1$, $y=1{,}2$, $\Delta x=0{,}016$.

3100. $u=p-\dfrac{qr}{p}+\sqrt{p+q+r}$. Naći $d_p u$ za $p=1$, $q=3$, $r=5$, $\Delta p=0{,}01$.

U zadacima 3101—3109 naći totalne diferencijale datih funkcija

3101. $z=x^2y^4-x^3y^3+x^4y^3$.

3102. $z=\dfrac{1}{2}\ln(x^2+y^2)$.

3103. $z=\dfrac{x+y}{x-y}$.

3104. $z=\arcsin\dfrac{x}{y}$.

3105. $z=\sin(xy)$.

3106. $z=\operatorname{arctg}\dfrac{x+y}{1-xy}$.

3107. $z=\dfrac{x^2+y^2}{x^2-y^2}$.

3108. $z=\operatorname{arctg}(xy)$.

3109. $u=x^{yz}$.

§ 4. Diferenciranje funkcija

Posredna funkcija

3124. $u = e^{x-2y}$, pri čemu je $x = \sin t$, $y = t^3$; $\dfrac{du}{dt} = ?$

3125. $u = z^2 + y^2 + zy$, $z = \sin t$, $y = e^t$; $\dfrac{du}{dt} = ?$

3126. $z = \arcsin(x - y)$, $x = 3t$, $y = 4t^3$; $\dfrac{dz}{dt} = ?$

3127. $z = x^2 y - y^2 x$, gde je $x = u \cos v$, $y = u \sin v$; $\dfrac{\partial z}{\partial u} = ?$ $\dfrac{\partial z}{\partial v} = ?$

3128. $z = x^2 \ln y$, $x = \dfrac{u}{v}$, $y = 3u - 2v$ $\dfrac{\partial z}{du} = ?$ $\dfrac{\partial z}{\partial v} = ?$

3129. $u = \ln(e^x - e^y)$; $\dfrac{\partial u}{\partial x} = ?$ Naći $\dfrac{du}{dx}$, Ako je $y = x^3$.

3130. $z = \operatorname{arctg}(xy)$; naći $\dfrac{dz}{dx}$, ako je $y = e^x$.

3131. $u = \arcsin \dfrac{x}{z}$, gde je $z = \sqrt{x^2 + 1}$; $\dfrac{du}{dx} = ?$

3132. $z = \operatorname{tg}(3t + 2x^2 - y)$, $x = \dfrac{1}{t}$, $y = \sqrt{t}$; $\dfrac{dz}{dt} = ?$

3133. $u = \dfrac{e^{ax}(x - z)}{a^2 + 1}$, $y = a \sin x$, $z = \cos x$; $\dfrac{du}{dx} = ?$

3134. $z = \dfrac{xy \operatorname{arctg}(xy + x + y)}{x + y}$; $dz = ?$

3135. $z = (x^2 + y^2)\, e^{\frac{x^2+y^2}{xy}}$; $\dfrac{\partial z}{\partial x} = ?$ $\dfrac{\partial z}{\partial y} = ?$ $dz = ?$

3136. $z = f(x^2 - y^2, e^{xy})$; $\dfrac{\partial z}{\partial x} = ?$ $\dfrac{\partial z}{\partial y} = ?$

3137. Uveriti se da funkcija $z = \operatorname{arctg} \dfrac{x}{y}$, u kojoj je $x = u + v$, $y = u - v$, zadovoljava relaciju

$$\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{u - v}{v^2 + u^2}.$$

3138. Uveriti se da funkcija $z = \varphi(x^2 + y^2)$, u kojoj je φ diferencijabilna funkcija, zadovoljava relaciju:

$$y \frac{\partial z}{\partial x} - x \frac{\partial z}{\partial y} = 0 .$$

3139. $u = \sin x + F(\sin y - \sin x)$; uveriti se da je $\dfrac{\partial u}{\partial y} \cos x + \dfrac{\partial u}{\partial x} \cos y = \cos x \cos y$, ma kakva bila diferencijabilna funkcija F.

3140. $z = \dfrac{y}{f(x^2 - y^2)}$, uveriti se da je $\dfrac{1}{x} \dfrac{\partial z}{\partial x} + \dfrac{1}{y} \dfrac{\partial z}{\partial y} = \dfrac{y}{y^2}$, ma kakva bila diferencijabilna funkcija f.

3141. Pokazati da homogena diferencijabilna funkcija $z=F\left(\frac{y}{x}\right)$ nultog stepena homogenosti (vidi zad. 2961) zadovoljava relaciju $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$.

3142. Pokazati da homogena funkcija $u=x^k F\left(\frac{z}{x};\ \frac{y}{x}\right)$, k-tog stepena homogenosti, u kojoj je F diferencijabilna funkcija, zadovoljava relaciju

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=ku.$$

3143. Proveriti tvrđenje formulisano u zadatku 3142 na funkciji

$$u=x^5\sin\frac{z^2+y^2}{x^2}.$$

3144. Neka je funkcija $f(x, y)$ diferencijabilna. Dokazati da, ako se promenljive x i y zamene linearnim homogenim funkcijama promenljivih X i Y, onda je tako dobijena funkcija $F(X, Y)$ vezana sa funkcijom $f(x, y)$ sledećom relacijom:

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}=X\frac{\partial F}{\partial X}+Y\frac{\partial F}{\partial Y}.$$

§ 5. Izvodi višeg reda

3181. $z=x^3+xy^2-5xy^3+y^5$. Uveriti se da je: $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$.

3182. $z=x^y$. Uveriti se da je $\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}$.

3183. $z=e^x(\cos y+x\sin y)$. Uveriti se da je

$$\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial^2 z}{\partial y\partial x}.$$

3184. $z=\operatorname{arctg}\frac{y}{x}$. Uveriti se da je $\frac{\partial^3 z}{\partial y^2\partial x}=\frac{\partial^3 z}{\partial x\partial y^2}$.

U zadacima 3185—3192 naći $\frac{\partial^2 z}{\partial x^2}$, $\frac{\partial^2 z}{\partial x\partial y}$, i $\frac{\partial^2 z}{\partial y^2}$ za date frnkcije.

3185. $z=\frac{1}{3}\sqrt{(x^2+y^2)^3}$.

3186. $z=\ln(x+\sqrt{x^2+y^2})$.

3187. $z=\operatorname{arctg}\frac{x+y}{1-xy}$.

3188. $z=\sin^2(ax+by)$.

3189. $z=e^{xe^y}$.

3190. $z=\frac{x-y}{x+y}$.

3191. $z=y^{\ln x}$.

3192. $z=\arcsin(xy)$.

3193. $u=\sqrt{x^2+y^2+z^2-2xz}$; $\quad \frac{\partial^2 u}{\partial y\partial z}=?$

3194. $z=e^{xy^2}$; $\quad \frac{\partial^3 z}{\partial x^2\partial y}=?$

3195. $s=\ln(x^2+y^2)$; $\quad \frac{\partial^3 z}{\partial x\partial y^2}=?$

3196. $z=\sin xy$; $\quad \frac{\partial^3 z}{\partial x\partial y^2}=?$

3197. $w=e^{xyz}$; $\quad \frac{\partial^3 w}{\partial x\partial y\partial z}=?$

3198. $v=x^m y^n z^p$; $\quad \frac{\partial^6 v}{\partial x\partial y^3\partial z^2}=?$

3199. $z=\ln(e^x+e^y)$; uveriti se da je $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=1$ i da je

$$\frac{\partial^2 z}{\partial x^2}\,\frac{\partial^2 z}{\partial y^2}-\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2=0.$$

3200. $u=e^x(x\cos y-y\sin y)$. Pokazati da je $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

3201. $u=\ln\frac{1}{\sqrt{x^2+y^2}}$; pokazati da je $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0.$

3202. $u=\frac{1}{\sqrt{x^2+y^2+z^2}}$; pokazati da je $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$

3203. $r=\sqrt{x^2+y^2+z^2}$; pokazati da je

$$\frac{\partial^2 r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}+\frac{\partial^2 r}{\partial z^2}=\frac{2}{r},\quad \frac{\partial^2(\ln r)}{\partial x^2}+\frac{\partial^2(\ln r)}{\partial y^2}+\frac{\partial^2(\ln r)}{\partial z^2}=\frac{1}{r^2}.$$

3204. Za koje vrednosti konstante a funkcija $v=x^3+axy^2$ zadovoljava jednačinu

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0?$$

3205. $z=\frac{y}{y^2-a^2x^2}$; pokazati da je $\frac{\partial^2 z}{\partial x^2}=a^2\frac{\partial^2 z}{\partial y^2}.$

3206. $v=\frac{1}{x-y}+\frac{1}{y-z}+\frac{1}{z-x}$; uveriti se da je

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}+2\left(\frac{\partial^2 v}{\partial x\,\partial y}+\frac{\partial^2 v}{\partial y\,\partial z}+\frac{\partial^2 v}{\partial z\,\partial x}\right)=0.$$

3207. $z=f(x, y)$, $\xi=x+y$, $\eta=x-y$; uveriti se da je

$$\frac{\partial^2 z}{\partial x^2}-\frac{\partial^2 z}{\partial y^2}=4\frac{\partial^2 z}{\partial\xi\,\partial\eta}.$$

3208. $v=x\ln(x+r)-r$, gde je $r^2=x^2+y^2$. Uveriti se da je

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=\frac{1}{x+r}.$$

3209. Izvesti obrazac za drugi izvod $\frac{d^2y}{dx^2}$ funkcije y, definisane implicitno jednačinom $f(x, y)=0$.

3210. $y=\varphi(x-at)+\psi(x+at)$. Pokazati da je

$$\frac{\partial^2 y}{\partial t^2}=a^2\frac{\partial^2 y}{\partial x^2},$$

ma kakve bile dvaput diferencijabilne funkcije φ i ψ.

3211. $u=\varphi(x)+\psi(y)+(x-y)\psi'(y)$. Uveriti se da je

$$(x-y)\frac{\partial^2 y}{\partial x\,\partial y}=\frac{\partial u}{\partial y}$$

(φ i ψ su dvaput diferencijabilne funkcije).

3212. $z=y\varphi(x^2-y^2)$. Uveriti se da je

$$\frac{1}{x}\frac{\partial z}{\partial x}+\frac{1}{y}\frac{\partial z}{\partial y}=\frac{z}{y^2}$$

(φ je diferencijabilna funkcija).

3213. $r=x\varphi(x+y)+y\psi(x+y)$; pokazati da je

$$\frac{\partial^2 r}{\partial x^2}-2\frac{\partial^2 r}{\partial x\partial y}+\frac{\partial^2 r}{\partial y^2}=0$$

(φ i ψ su dvaput diferencijabilne funkcije).

3214. $u=\frac{1}{y}[\varphi(ax+y)+\psi(ax-y)]$. Pokazaii da je

$$\frac{\partial^2 u}{\partial x^2}=\frac{a^2}{y^2}\cdot\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right).$$

3215. $u=\frac{1}{x}[\varphi(x-y)+\psi(x+y)]$. Pokazati da je

$$\frac{\partial}{\partial x}\left(x^2\frac{\partial u}{\partial x}\right)=x^2\frac{\partial^2 u}{\partial y^2}.$$

3216. $u=xe^y+ye^x$. Pokazati da je

$$\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 y}{\partial y^3}=x\frac{\partial^3 u}{\partial x\partial y^2}+y\frac{\partial^3 u}{\partial x^2\partial y}.$$

3217. $u=e^{xyz}$. Pokazati da je

$$\frac{\partial^3 y}{\partial x\partial y\partial z}=xy\frac{\partial^2 u}{\partial x\partial y}+2x\frac{\partial u}{\partial x}+u.$$

3218. $u=\ln\frac{x^2-y^2}{xy}$. Pokazati da je

$$\frac{\partial^3 u}{\partial x^3}+\frac{\partial^3 u}{\partial x^2\partial y}-\frac{\partial^3 u}{\partial x\partial y^2}-\frac{\partial^3 u}{\partial y^3}=2\left(\frac{1}{y^3}-\frac{1}{x^3}\right).$$

U zadacima 3219—3224 naći diferencijale drugog reda za date funkcije.

3219. $z=xy^2-x^2y$. **3220.** $z=\ln(x-y)$.

3221. $z=\frac{1}{2(x^2+y^2)}$. **3222.** $z=x\sin^2 y$.

3223. $z=e^{xy}$. **3224.** $u=xyz$.

3225. $z=\sin(2x+y)$. Naći d^3z u tačkama $(0,\pi)$; $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

3226. $u+\sin(x+y+z)$; $d^2u=?$

3227. $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$; $d^2z=?$

3228. $z^3-3xyz=a^3$; $d^2z=?$

3229. $3x^2y^2+2z^2xy-2zx^3+4zy^3-4=0$. Naći d^2z u tački $(2, 1, 2)$.

Rješenja

3032. $\frac{1}{v}\frac{\partial v}{\partial T}$ za $T=T_0$.

3033. $\frac{\partial \theta}{\partial t}$ — brzina menjanja temperature u datoj tački; $\frac{\partial \theta}{\partial x}$ — brzina menjanja temperature u odnosu na dužinu (duž štapa), u datom trenutku vremena.

3034. $\frac{\partial S}{\partial h}=b$ — brzina menjanja površine u zavisnosti od visine pravougaonika; $\frac{\partial S}{\partial h}=h$ — brzina menjanja površine u zavisnosti od osnovice pravougaonika.

3036. $\frac{\partial z}{\partial x}=1,\ \frac{\partial z}{\partial y}=-1.$ **3037.** $\frac{\partial z}{\partial x}=3x^2y-y^3;\ \frac{\partial z}{\partial y}=x^3-3y^2x.$

3038. $\frac{\partial \theta}{\partial x}=ae^{-t};\ \frac{\partial \theta}{\partial t}=-axe^{-t}+b.$

3039. $\frac{\partial z}{\partial u}=\frac{1}{v}-\frac{v}{u^2};\ \frac{\partial z}{\partial v}=-\frac{u}{v^2}+\frac{1}{u}.$

3040. $\frac{\partial z}{\partial x}=\frac{x^4+3x^2y^2-2xy^3}{(x^2+y^2)^2};$

$\frac{\partial z}{\partial y}=\frac{y^4+3x^2y^2-2x^3y}{(x^2+y^2)^2}.$

3041. $\frac{\partial z}{\partial x}=30xy\,(5x^2y-y^3+7)^2;$

$\frac{\partial z}{\partial y}=3\,(5x^2y-y^3+7)^2\,(5x^2-3y^2).$

3042. $\frac{\partial z}{\partial x}=\sqrt{y}-\frac{y}{3\sqrt[3]{x^4}};\ \frac{\partial z}{\partial y}=\frac{x}{2\sqrt{y}}+\frac{1}{\sqrt[3]{x}}.$

3043. $\frac{\partial z}{\partial x}=\frac{1}{\sqrt{x^2+y^2}};\ \frac{\partial z}{\partial y}=\frac{y}{x^2+y^2+x\sqrt{x^2+y^2}}.$

3044. $\frac{\partial z}{\partial x}=\frac{y}{x^2+y^2};\ \frac{\partial z}{\partial y}=-\frac{x}{x^2+y^2}.$

3045. $\frac{\partial z}{\partial x}=\frac{y}{(x^2+y^2)\left(\operatorname{arctg}\frac{y}{x}\right)^2};\ \frac{\partial z}{\partial y}=-\frac{x}{(x^2+y^2)\left(\operatorname{arctg}\frac{y}{x}\right)^2}.$

3046. $\frac{\partial z}{\partial x}=yx^{y-1};\ \frac{\partial z}{\partial y}=x^y\ln x.$ **3047.** $\frac{\partial z}{\partial x}=\frac{2x}{x^2+y^2};\ \frac{\partial z}{\partial y}=\frac{2y}{x^2+y^2}.$

3048. $\frac{\partial z}{\partial x}=-\frac{2}{\sqrt{x^2+y^2}};\ \frac{\partial z}{\partial y}=\frac{2x}{y\sqrt{x^2+y^2}}.$

3049. $\frac{\partial z}{\partial x}=\frac{xy\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}};\ \frac{\partial z}{\partial y}=-\frac{x^2\sqrt{2}}{(x^2+y^2)\sqrt{x^2-y^2}}.$

3050. $\frac{\partial z}{\partial x}=\frac{2}{y\sin\frac{2x}{y}};\ \frac{\partial z}{\partial y}=-\frac{2x}{y^2\sin\frac{2x}{y}}.$

3051. $\frac{\partial z}{\partial x}=-\frac{1}{y}e^{-\frac{x}{y}};\ \frac{\partial z}{\partial y}=\frac{x}{y^2}e^{-\frac{x}{y}}.$

3052. $\frac{\partial z}{\partial x}=\frac{1}{x+\ln y}$; $\quad\frac{\partial z}{\partial y}=\frac{1}{y(x+\ln y)}$.

3053. $\frac{\partial u}{\partial v}=-\frac{w}{v^2+w^2}$; $\quad\frac{\partial u}{\partial w}=\frac{v}{v^2+w^2}$.

3054. $\frac{\partial z}{\partial x}=\frac{1}{y}\cos\frac{x}{y}\cos\frac{y}{x}+\frac{y}{x^2}\sin\frac{x}{y}\sin\frac{y}{x}$;

$\frac{\partial z}{\partial y}=-\frac{x}{y^2}\cos\frac{x}{y}\cos\frac{y}{x}-\frac{1}{x}\sin\frac{x}{y}\sin\frac{y}{x}$.

3055. $\frac{\partial z}{\partial x}=\frac{y}{x^2}3^{-\frac{y}{x}}\ln 3$; $\quad\frac{\partial z}{\partial y}=-\frac{1}{x}{}^{-\frac{y}{x}}\ln 3$.

3056. $\frac{\partial z}{\partial x}=y^2(1+xy)^{y-1}$; $\quad\frac{\partial z}{\partial y}=xy(1+xy)^{y-1}+(1+xy)^y\ln(1+xy)$.

3057. $\frac{\partial z}{\partial x}=y\ln(x+y)+\frac{xy}{x+y}$; $\quad\frac{\partial z}{\partial y}=x\ln(x+y)+\frac{xy}{x+y}$.

3058. $\frac{\partial z}{\partial x}=x^{x^y}x^{y-1}(y\ln x+1)$; $\quad\frac{\partial z}{\partial y}=x^y x^{x^y}\ln^2 x$

3059. $\frac{\partial u}{\partial x}=yz$; $\quad\frac{\partial u}{\partial y}=xz$; $\quad\frac{\partial u}{\partial z}=xy$.

3060. $\frac{\partial u}{\partial x}=y+z$; $\quad\frac{\partial u}{\partial y}=x+z$; $\quad\frac{\partial u}{\partial z}=x+y$.

3061. $\frac{\partial u}{\partial x}=\frac{x}{\sqrt{x^2+y^2+z^2}}$; $\quad\frac{\partial u}{\partial y}=\frac{y}{\sqrt{x^2+y^2+z^2}}$; $\quad\frac{\partial u}{\partial z}=\frac{z}{\sqrt{x^2+y^2+z^2}}$.

3062. $\frac{\partial u}{\partial x}=3x^2+3y-1$; $\quad\frac{\partial u}{\partial y}=z^2+3x$; $\quad\frac{\partial u}{\partial z}=2yz+1$.

3063. $\frac{\partial w}{\partial x}=yz+vz+vy$; $\quad\frac{\partial w}{\partial y}=xz+zv+vx$; $\quad\frac{\partial w}{\partial z}=xy+yv+vx$; $\quad\frac{\partial w}{\partial v}=yz+xz+xy$.

3064. $\frac{\partial u}{\partial x}=(3x^2+y^2+z^2)e^{x(x^2+y^2+z^2)}$;

$\frac{\partial u}{\partial y}=2xye^{x(x^2+y^2+z^2)}$; $\quad\frac{\partial u}{\partial z}=2xze^{x(x^2+y^2+z^2)}$.

3065. $\frac{\partial u}{\partial x}=2x\cos(x^2+y^2+z^2)$; $\quad\frac{\partial u}{\partial y}=2y\cos(x^2+y^2+z^2)$;

$\frac{\partial u}{\partial z}=2z\cos(x^2+y^2+z^2)$.

3066. $\frac{\partial u}{\partial x}=\frac{\partial u}{\partial y}=\frac{\partial u}{\partial z}=\frac{1}{x+y+z}$.

3067. $\frac{\partial u}{\partial x}=\frac{y}{z}x^{\frac{y}{z}-1}$; $\quad\frac{\partial u}{\partial y}=\frac{1}{z}x^{\frac{y}{z}}\ln x$; $\quad\frac{\partial u}{\partial z}=-\frac{y}{z^2}x^{\frac{y}{z}}\ln x$.

3068. $\frac{\partial u}{\partial x}=y^z x^{y^z-1}$; $\quad\frac{\partial u}{\partial y}=zy^{z-1}x^{y^z}\ln x$; $\quad\frac{\partial u}{\partial z}=y^z x^{y^z}\ln x\ln y$.

3069. $\frac{2}{5}$, $\frac{1}{5}$.

3070. 0, $\frac{1}{4}$.

3071. $\frac{\partial z}{\partial x}=2(2x+y)^{2x+y}[1+\ln(2x+y)]$;

$\frac{\partial z}{\partial y}=(2x+y)^{2x+y}[1+\ln(2x+y)]$.

3072. $\frac{\partial z}{\partial x}=\frac{3}{x\ln y}\left(1+\frac{\ln x}{\ln y}\right)^2$; $\frac{\partial z}{\partial y}=-\frac{3\ln x}{y\ln^2 y}\left(1+\frac{\ln x}{\ln y}\right)^2$.

3073. $\frac{\partial z}{\partial x}=ye^{\sin\pi xy}(1+\pi xy\cos\pi xy)$;

$\frac{\partial z}{\partial y}=xe^{\sin\pi xy}(1+\pi xy\cos\pi xy)$.

3074. $\frac{\partial z}{\partial x}=\frac{1-x^2-y^2-\sqrt{x^2+y^2}}{(1+\sqrt{x^2+y^2})^2}\,2x$;

$\frac{\partial z}{\partial y}=\frac{1-x^2-y^2-\sqrt{x^2+y^2}}{(1+\sqrt{x^2+y^2})^2}\,2y$.

3075. $\frac{\partial z}{\partial x}=\frac{y\sqrt{x^y}}{2x(1+x^y)}$; $\frac{\partial z}{\partial y}=\frac{\sqrt{x^y}\ln x}{2(1+x^y)}$.

3076. $\frac{\partial z}{\partial x}=-\frac{y}{(1+\sqrt{xy})\sqrt{xy-x^2y^2}}$; $\frac{\partial z}{\partial y}=-\frac{x}{(1+\sqrt{xy})\sqrt{xy-x^2y^2}}$.

3077. $\frac{\partial z}{\partial x}=\frac{y^2+2xy}{\sqrt{1+(xy^2+yx^2)^2}}$; $\frac{\partial z}{\partial y}=\frac{x^2+2xy}{\sqrt{1+(xy^2+yx^2)^2}}$.

3078. $\frac{\partial z}{\partial x}=-\frac{1}{x^2}\sqrt{\frac{xy-x-y}{xy+x+y}}$; $\frac{\partial z}{\partial y}=-\frac{1}{y^2}\sqrt{\frac{xy-x-y}{xy+x+y}}$.

3079. $\frac{\partial z}{\partial x}=\frac{y\left[\left(1+\operatorname{arctg}^2\frac{y}{x}\right)^2+2\operatorname{arctg}^3\frac{y}{x}\right]}{(x^2+y^2)\left(1+\operatorname{arctg}^2\frac{y}{x}\right)\left(1+\operatorname{arctg}\frac{y}{x}\right)^2}$;

$\frac{\partial z}{\partial y}=-\frac{x\left[\left(1+\operatorname{arctg}^2\frac{y}{x}\right)^2+2\operatorname{arctg}^3\frac{y}{x}\right]}{(x^2+y^2)\left(1+\operatorname{arctg}^2\frac{y}{x}\right)\left(1+\operatorname{arctg}\frac{y}{x}\right)^2}$.

3080. $\frac{\partial u}{\partial x}=-\frac{4kx}{(x^2+y^2+z^2)^3}$;

$\frac{\partial u}{\partial y}=-\frac{4ky}{(x^2+y^2+z^2)^3}$;

$\frac{\partial u}{\partial z}=-\frac{4kz}{(x^2+y^2+z^2)^3}$.

3081. $\frac{\partial u}{\partial x}=\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$; $\frac{\partial u}{\partial y}=-\frac{z(x-y)^{z-1}}{1+(x-y)^{2z}}$; $\frac{\partial u}{\partial z}=\frac{(x-y)^z\ln(x-y)}{1+(x-y)^{2z}}$.

3082. $\frac{\partial u}{\partial x}=yz(\sin x)^{yz-1}\cos x$; $\frac{\partial u}{\partial y}=z(\sin x)^{yz}\ln\sin x$;

$\frac{\partial u}{\partial z}=y(\sin x)^{yz}\ln\sin x$.

3083. $\frac{\frac{\partial u}{\partial x}}{x}=\frac{\frac{\partial u}{\partial y}}{y}=\frac{\frac{\partial u}{\partial z}}{z}=\frac{2}{r(r^2-1)}$, gde $r=\sqrt{x^2+y^2+z^2}$.

3084. $\frac{\partial w}{\partial x}=(2xy^2-yzv)\operatorname{tg}^3\alpha$; $\frac{\partial w}{\partial y}=(2x^2y-xzv)\operatorname{tg}^3\alpha$; $\frac{\partial w}{\partial z}=(2zv^2-xyv)\operatorname{tg}^3\alpha$;

$\frac{\partial w}{\partial v}=(2z^2v-xyz)\operatorname{tg}^3\alpha$, gde je $\alpha=x^2y^2+z^2v^2-xyzv$.

3085. 4. 3086. $\left(\dfrac{\partial u}{\partial z}\right)_{\substack{z=b\\ t=a}}=\dfrac{3b}{2}\sqrt{\dfrac{ab}{b^2-a^2}}$;

$$\left(\frac{\partial u}{\partial t}\right)_{\substack{z=b\\ t=a}}=-\frac{3a}{3}\sqrt{\frac{ab}{b^2-a^2}};$$

3087. 1 і −1. 3088. $\dfrac{\sqrt{2}}{2}$. 3089. $\dfrac{3}{2}$. 3090. $\dfrac{13}{22}$. 3091. 45°.

3092. 30°. 3093. $\operatorname{arctg}\dfrac{4}{7}$.

3094. $d_x z=(y^3-6xy^2)\,dx$; $d_y z=(3xy^2-6x^2y+8y^3)\,dy$.

3095. $d_x z=\dfrac{x\,dx}{\sqrt{x^2+y^2}}$; $d_y z=\dfrac{y\,dy}{\sqrt{x^2+y^2}}$.

3096. $d_x z=\dfrac{y(y^2-x^2)\,dx}{(x^2+y^2)^2}$; $d_y z=\dfrac{x(x^2-y^2)\,dy}{(x^2+y^2)^2}$.

3097. $d_x u=\dfrac{3x^2\,dx}{x^3+2y^3-z^3}$; $d_y u=\dfrac{6y^3\,dy}{x^3+2y^3-z^3}$; $d_z u=\dfrac{-3z^2\,dz}{x^3+2y^3-z^3}$.

3098. $\dfrac{1}{270}$. 3099. $\approx 0{,}0187$. 3100. $\dfrac{97}{600}$.

3101. $xy\left[(2y^3-3xy^2+4x^2y)\,dx+(4y^2x-3yx^2+2x^3)\,dy\right]$.

3102. $\dfrac{x\,dx+y\,dy}{x^2+y^2}$. 3103. $\dfrac{2(x\,dy-y\,dx)}{(x-y)^2}$. 3104. $\dfrac{y\,dx-x\,dy}{y\sqrt{y^2-x^2}}$.

3105. $(x\,dy+y\,dx)\cos(xy)$. 3106. $\dfrac{dx}{1+x^2}+\dfrac{dy}{1+y^2}$.

3107. $\dfrac{4xy(x\,dy-y\,dx)}{(x^2-y^2)^2}$. 3108. $\dfrac{x\,dy+y\,dx}{1+x^2y^2}$.

3109. $x^{xy-1}(yz\,dx+zx\ln x\,dx+xy\ln x\,dz)$,

3124. $e^{\sin t-2t^3}(\cos t-6t^2)$. **3125.** $\sin 2t+2e^{2t}+e^t(\sin t+\cos t)$.

3126. $\dfrac{3-12t^2}{\sqrt{1-(3t-4t^3)^2}}$. **3127.** $\dfrac{\partial z}{\partial u}=3u^2\sin v\cos v(\cos v-\sin v)$;

$$\frac{\partial z}{\partial v}=u^3(\sin v+\cos v)(1-3\sin v\cos v).$$

3128. $\dfrac{\partial z}{\partial u}=2\dfrac{u}{v^2}\ln(3u-2v)+\dfrac{3u^2}{v^2(3u-2v)}$;

$$\frac{\partial z}{\partial v}=-\frac{2u^2}{v^3}\ln(3u-2v)-\frac{2u^2}{v^2(3u-2v)}.$$

3129. $\dfrac{\partial u}{\partial x}=\dfrac{e^x}{e^x+e^y}$; $\dfrac{du}{dx}=\dfrac{e^x+3e^{x^3}x^2}{e^x+e^{x^3}}$.

3130. $\dfrac{dz}{dx}=\dfrac{e^x(x+1)}{1+x^2e^{2x}}$. **3131.** $\dfrac{du}{dx}=\dfrac{1}{1+x^2}$.

3132. $\dfrac{dz}{dt}=\left(3-\dfrac{4}{t^3}-\dfrac{1}{2\sqrt{t}}\right)\sec^2\left(3t+\dfrac{2}{t^2}-\sqrt{t}\right)$.

3133. $\dfrac{du}{dx}=e^{ax}\sin x$. **3134.** $dz=\dfrac{y^2dx+x^2dy}{(x+y)^2}\operatorname{arctg}(xy+x+y)+\dfrac{xy[(y+1)dx+(x+1)dy]}{(x+y)[1+(xy+x+y)^2]}$.

3135. $\dfrac{e^{\frac{x^2+y^2}{xy}}}{x^2y^2}[(y^4-x^4+2xy^3)x\,dy+(x^4-y^4+2x^3y)y\,dx]$.

3136. $$\left.\begin{aligned}\frac{\partial z}{\partial x}&=2x\frac{\partial f}{\partial u}+ye^{xy}\frac{\partial f}{\partial v}.\\ \frac{\partial z}{\partial y}&=-2y\frac{\partial f}{\partial u}+xe^{xy}\frac{\partial f}{\partial v}\end{aligned}\right\}\begin{aligned}u&=x^2-y^2;\\ v&=e^{xy}.\end{aligned}$$

3185. $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{2x^2+y^2}{\sqrt{x^2+y^2}}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{x^2+2y^2}{\sqrt{x^2+y^2}}$; $\dfrac{\partial^2 z}{\partial x\partial y}=\dfrac{xy}{\sqrt{x^2+y^2}}$.

3186. $\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{x}{(x^2+y^2)^{\frac{3}{2}}}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{x^3+(x^2-y^2)\sqrt{x^2+y^2}}{(x^2+y^2)^{\frac{3}{2}}(x+\sqrt{(x^2+y^2)^2}}$;

$$\frac{\partial^2 z}{\partial x\partial y}=-\frac{y}{(x^2+y^2)^{\frac{3}{2}}}.$$

3187. $\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{2x}{(1+x^2)^2}$; $\dfrac{\partial^2 z}{\partial y^2}=-\dfrac{2y}{(1+y^2)^2}$; $\dfrac{\partial^2 z}{\partial x\partial y}=0$.

3188. $\dfrac{\partial^2 z}{\partial x^2}=2a^2\cos 2(ax+by)$; $\dfrac{\partial^2 z}{\partial y^2}=2b^2\cos 2(ax+by)$;

$$\frac{\partial^2 z}{\partial x\partial y}=2ab\cos 2(ax+by).$$

3189. $\dfrac{\partial^2 z}{\partial x^2}=e^{xe^y+2y}$; $\dfrac{\partial^2 z}{\partial y^2}=x(1+xe^y)e^{xe^y+y}$; $\dfrac{\partial^2 z}{\partial x\partial y}=(1+xe^y)e^{xe^y+y}$.

3190. $\dfrac{\partial^2 z}{\partial x^2}=-\dfrac{4y}{(x+y)^3}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{4x}{(x+y)^3}$; $\dfrac{\partial^2 z}{\partial x\,\partial y}=\dfrac{2(x-y)}{(x+y)^3}$.

3191. $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{\ln y(\ln y+1)}{x^2}e^{\ln x\ln y}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\ln x(\ln x-1)}{y^2}e^{\ln x\ln y}$;

$\dfrac{\partial^2 z}{\partial x\,\partial y}=\dfrac{\ln x\ln y+1}{xy}e^{\ln x\ln y}$.

3192. $\dfrac{\partial^2 z}{\partial x^2}=\dfrac{xy^3}{\sqrt{(1-x^2y^2)^3}}$; $\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial^3 y}{\sqrt{(1-x^2y^2)^3}}$; $\dfrac{\partial^2 z}{\partial x\,\partial y}=\dfrac{1}{\sqrt{(1-x^2y^2)^3}}$.

3193. $\dfrac{(x-z)y}{\sqrt{(x^2+y^2+z^2-2xz)^3}}$. 3194. $2y^3(2+xy^2))e^{xy^2}$.

3195. $\dfrac{4x(3y^2-x^2)}{(x^2+y^2)^3}$. 3196. $-x(2\sin xy+xy\cos xy)$.

3197. $(x^2y^2z^2+3xyz+1)e^{xyz}$.

3198. $mn(n-1)(n-2)p(p-1)x^{m-1}y^{n-3}z^{p-2}$. 3204. $a=-3$.

3209. $$\frac{d^2y}{dx^2}=-\frac{\dfrac{\partial^2 f}{\partial x^2}\left(\dfrac{\partial f}{\partial y}\right)^2-2\dfrac{\partial^2 f}{\partial x\,\partial y}\dfrac{\partial f}{\partial x}\dfrac{\partial f}{\partial y}+\dfrac{\partial^2 f}{\partial y^2}\left(\dfrac{\partial f}{\partial x}\right)^2}{\left(\dfrac{\partial f}{\partial y}\right)^3}=\frac{1}{\left(\dfrac{\partial f}{\partial y}\right)^3}\begin{vmatrix} 0 & \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial^2 f}{\partial x^2} & \dfrac{\partial^2 f}{\partial x\,\partial y} \\ \dfrac{\partial f}{\partial y} & \dfrac{\partial^2 f}{\partial x\,\partial y} & \dfrac{\partial^2 f}{\partial y^2}\end{vmatrix}$$

3219. $-2y\,dx^2+4(y-x)\,dx\,dy+2x\,dy^2$. 3220. $-\dfrac{(dx-dy)^2}{(x-y)^2}$.

3221. $\dfrac{(3x^2-y^2)\,dx^2+8xy\,dx\,dy+(3y^2-x^2)\,dy^2}{(x^2+y^2)^3}$.

3222. $2\sin 2y\,dx\,dy+2x\cos 2y\,dy^2$. 3223. $e^{xy}[(y\,dx+y\,dy)^2+2\,dx\,dy]$.

3224. $2(z\,dx\,dy+y\,dx\,dz+x\,dy\,dz)$.

3225. $-\cos(2x+y)(2\,dx+dy)^3$; $(2\,dx+dy)^3$; 0.

3226. $-\sin(x+y+z)(dx+dy+dz)^2$.

3227. $-\dfrac{c^4}{z^3}\left[\left(\dfrac{x^2}{a^2}+\dfrac{z^2}{c^2}\right)\dfrac{dx^2}{a^2}+\dfrac{2xy}{a^2b^2}dx\,dy+\left(\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)\dfrac{dy^2}{b^2}\right]$.

3228. $\dfrac{2z[xy^3\,dx^2+(x^2y^2+2xyz^2-z^4)\,dx\,dy+x^3y\,dy^2]}{(z^2-xy)^3}$.

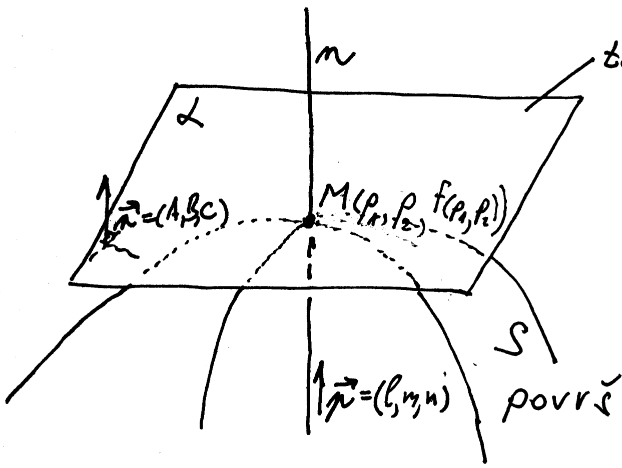
3229. $-31{,}5\,dx^2+206\,dx\,dy-306\,dy^2$. 3230. $\dfrac{d^2y}{dt^2}+y$.

Jednačina tangentne ravni i jednačina normale na površ

Jednačina tangentne ravni (hiperravni) na površ S, čija je jednačina $z=f(x_1,x_2)$, u tački $M(\rho_1,\rho_2,f(\rho_1,\rho_2))$ (ako je f diferencijabilna u tački (ρ_1,ρ_2)) glasi:

$$z-f(\rho_1,\rho_2)=f'_{x_1}(\rho_1,\rho_2)(x_1-\rho_1)+f'_{x_2}(\rho_1,\rho_2)(x_2-\rho_2)$$

Može li se uspostaviti sličnost sa jednačinom tangente na krivu liniju $y=f(x)$ u ravni?



$Ax+By+Cz+D=0$

$M(\rho_1,\rho_2,f(\rho_1,\rho_2))$ tačka dodira

n - normala na površ $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$

Jednačina normale na površ $z=f(x,y)$ u tački $M(\rho_1,\rho_2,f(\rho_1,\rho_2))$ (ako je f diferencijabilna u (ρ_1,ρ_2)) glasi:

$$\frac{x-\rho_1}{f'_x(\rho_1,\rho_2)}=\frac{y-\rho_2}{f'_y(\rho_1,\rho_2)}=\frac{z-f(\rho_1,\rho_2)}{-1}$$

sličnost sa krivom $y=f(x)$ u ravni: $k_1\cdot k_2=-1$, $M(\rho_1,\rho_2)$ $y-\rho_2=f'(\rho_1)(x-\rho_1)$

$k_2=\dfrac{-1}{k_1}$, $y-\rho_2=\dfrac{-1}{f'(\rho_1)}(x-\rho_1)$

$\dfrac{x-\rho_1}{f'(\rho_1)}=\dfrac{y-\rho_2}{-1}$

Ako površ S ima jednačinu u implicitnom obliku $F(x,y,z)=0$

$$\alpha:\ F'_x(\rho_1,\rho_2,f(\rho_1,\rho_2))(x-\rho_1)+F'_y(\rho_1,\rho_2,f(\rho_1,\rho_2))(y-\rho_2)+F'_z(\rho_1,\rho_2,f(\rho_1,\rho_2))(z-f(\rho_1,\rho_2))=0$$

$$n:\ \frac{x-\rho_1}{F'_x(\rho_1,\rho_2,f(\rho_1,\rho_2))}=\frac{y-\rho_2}{F'_y(\rho_1,\rho_2,f(\rho_1,\rho_2))}=\frac{z-f(\rho_1,\rho_2)}{F'_z(\rho_1,\rho_2,f(\rho_1,\rho_2))}$$

(#) Naći jednačinu tangentne ravni i normale na površ

a) $z=\frac{x^2}{2}-y^2$ u tački $M(2,-1,1)$

b) $3xyz-z^3=a^3$ u tački za koju je $x=0,\ y=a$

c) $z=x^2+2y^2$ u tački $A(1,1,3)$

d) $z=\operatorname{arc\,tg}\frac{y}{x}$ u tački $(1,1,\frac{\pi}{4})$ → rj. $\alpha:\ z=\frac{\pi}{4}-\frac{1}{2}(x-y)$
$n:\ \frac{x-1}{1}=\frac{y-1}{-1}=\frac{z-\frac{\pi}{4}}{2}$

e) $z=\sqrt{169-x^2-y^2}$ → rj. u tački $(3,4,z(3,4))$ $\alpha:\ 3x+4y+12z-169=0$
$n:\ \frac{x-3}{3}=\frac{y-4}{4}=\frac{z-12}{12}$

f) $\frac{x^2}{16}+\frac{y^2}{9}-\frac{z^2}{8}=0$ u tački $M(4,3,4)$ → rj. $\alpha:\ 3x+4y-6z=0$
$n:\ \frac{x-4}{3}=\frac{y-3}{4}=\frac{z-4}{-6}$

g) $x^2+y^2+z^2=2Rz$ u tački $(R\cos\alpha,\ R\sin\alpha,\ R)$ $(R>0)$.

Rj. a) $z=f(x,y)$, $z-f(p_1,p_2)=f'_x(p_1,p_2)(x-p_1)+f'_y(p_1,p_2)(y-p_2)$ jedn. tang. ravni

$z=\frac{x^2}{2}-y^2$, $z'_x=x$, $z'_x(2,-1)=2$, $\frac{\partial z}{\partial y}=-2y$, $z'_y(2,-1)=2$

$M(2,-1,1)$, $f(2,-1)=1$ $\qquad z-1=2(x-2)+2(y+1)$

$2x+2y-z-1$ jednačina tangentne ravni

$$\frac{x-p_1}{f'_x(p_1,p_2)}=\frac{y-p_2}{f'_y(p_1,p_2)}=\frac{z-f(p_1,p_2)}{-1}\Rightarrow\frac{x-2}{2}=\frac{y+1}{2}=\frac{z-1}{-1}$$ jedn. normale

b) Nađimo tačku dodira tangentne ravni i površi

$x=0,\ y=a$, $3xyz-z^3=a^3\Rightarrow -z^3=a^3\Rightarrow z=-a$

Tačka dodira je $M(0,a,-a)$

$F'_x=3yz\Rightarrow F'_x(0,a,-a)=-3a^2$

$F'_y=3xz\Rightarrow F'_y(0,a,-a)=0$

$F'_z=3xy-3z^2\Rightarrow F'_z(0,a,-a)=-3a^2$

$\alpha:\ F'_x(p_1,p_2,f(p_1,p_2))(x-p_1)+F'_y(p_1,p_2,f(p_1,p_2))(y-p_2)+F'_z(p_1,p_2,f(p_1,p_2))(z-f(p_1,p_2))=0$

$-3a^2(x-0)+0(y-a)+(-3a^2)(z-(-a))=0\Rightarrow -3a^2x-3a^2z-3a^3=0$

tj. $x+z+a=0$ jedn. tang. ravni

$$\frac{x-0}{-3a^2}=\frac{y-a}{0}=\frac{z+a}{-3a^2}\Rightarrow\frac{x}{1}=\frac{y-a}{0}=\frac{z+a}{1}$$ jednačina normale

c) rj. $\alpha:\ 2x+4y-z-3=0$
$n:\ \frac{x-1}{2}=\frac{y-1}{4}=\frac{z-3}{-1}$

g) rj. $\alpha:\ x\cos\alpha+y\sin\alpha-R=0$
$n:\ \frac{x-R\cos\alpha}{\cos\alpha}=\frac{y-R\sin\alpha}{\sin\alpha}=\frac{z-R}{0}$

Na površ $x^2+2y^2+3z^2=21$ postaviti tangentnu ravan paralelnu ravni $x+4y+6z=0$.

Rj.

$\beta: Ax+By+Cz+D=0$

$\beta: ?$ $\quad \alpha \parallel \beta$

$\alpha: x+4y+6z=0$

$\vec{n}_\alpha=(1,4,6), \quad \vec{n}_\beta \parallel \vec{n}_\alpha$

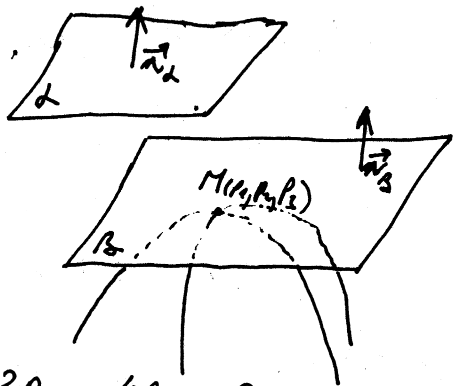
Treba nam tačka dodira tražene tangentne ravni sa površi $x^2+2y^2+3z^2=21$.

$$F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)=0$$

$F'_x=2x$

$F'_y=4y$

$F'_z=6z$



$$n: \frac{x-p_1}{F'_x(p_1,p_2,p_3)}=\frac{y-p_2}{F'_y(p_1,p_2,p_3)}=\frac{z-z_0}{F'_z(p_1,p_2,p_3)}$$

Vektor normale tražene tangentne ravni je

$\vec{n}_\beta=(2p_1, 4p_2, 6p_3)$

$$\vec{n}_\alpha \parallel \vec{n}_\beta \Rightarrow \frac{2p_1}{1}=\frac{4p_2}{4}=\frac{6p_3}{6} \Rightarrow 2p_1=p_2=p_3$$

odredimo p_1, p_2 i p_3

$p_1^2+2\cdot 4p_1^2+3\cdot 4p_1^2=21$

$21p^2=21$

$p_1=\pm 1 \Rightarrow p_2=p_3=\pm 2$

I rješenje:

$p_1=-1, \; p_2=p_3=-2$

$-2(x+1)-8(y+2)-12(z+2)=0$

$-2x-8y-12z=42$

$x+4y+6z=-21$

II rješenje $p_1=1, \; p_2=p_3=2$

$2(x-1)+8(y-2)+12(z-2)=0$

$2x+8y+12z-42=0 \quad /:2$

$x+4y+6z=21$

jednačina tražene tangentne ravni

(#) Odrediti jednačine normale i jednačinu tangentne ravni površi $z=\sqrt{169-x^2-y^2}$ u tački $(3,4,z(3,4))$.

Rj:

$$z(3,4)=\sqrt{169-9-16}=\sqrt{144}=12$$

$M(3,4,12)$

jednačina tangentne ravni i normale na površ $z=f(x,y)$ u tački $M(p_1,p_2,p_3)$:
$$z-p_3=z'_x(p_1,p_2)(x-p_1)+z'_y(p_1,p_2)(y-p_2)$$

$$\frac{x-p_1}{z'_x(p_1,p_2)}=\frac{y-p_2}{z'_y(p_1,p_2)}=\frac{z-p_3}{-1}$$

$$=\frac{1}{2\sqrt{169-x^2-y^2}}(-2x)=\frac{-x}{\sqrt{169-x^2-y^2}}\Rightarrow z'_x(3,4)=\frac{-3}{\sqrt{169-25}}=\frac{-3}{12}=-\frac{1}{4}$$

$$\frac{\partial z}{\partial y}=\frac{1}{2\sqrt{169-x^2-y^2}}(-2y)=\frac{-y}{\sqrt{169-x^2-y^2}}\Rightarrow z'_y(3,4)=\frac{-4}{12}=-\frac{1}{3}$$

$$z-12=-\frac{1}{4}(x-3)-\frac{1}{3}(y-4)\quad /\cdot 12$$

$$12z-144=-3(x-3)-4(y-4)$$

$$3x+4y+12z-144-9-16=0$$

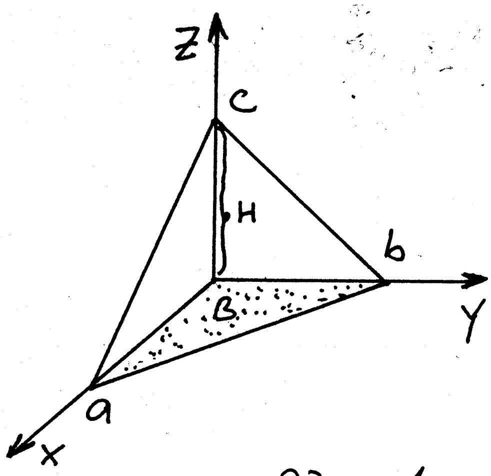
$$3x+4y+12z-169=0$$ jednačina tangentne ravni na površ z

$$\frac{x-3}{-\frac{1}{4}}=\frac{y-4}{-\frac{1}{3}}=\frac{z-12}{-1}\quad /\cdot\left(\frac{1}{-12}\right)$$

$$\frac{x-3}{3}=\frac{y-4}{4}=\frac{z-12}{12}$$ jednačina normale na površ z

(#) Dokazati da tangentne ravni površi $z=\frac{1}{xy}$ tvore s koordinatnim ravnima piramide konstantne zapremine.

Rj. Jednačina tangentne ravni na površ $z=f(x,y)$ u tački $M(\rho_1,\rho_2,\rho_3)$: $z-\rho_3 = z'_x(\rho_1,\rho_2)(x-\rho_1)+z'_y(\rho_1,\rho_2)(y-\rho_2)$



$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ kanonični oblik jednačine ravni gdje su a, b i c odsječci koje ravan odsjeca na koordinatnim osama

$$V_{piramide}=\frac{B\cdot H}{3}=\frac{\frac{a\cdot b}{2}\cdot c}{3}=\frac{a\cdot b\cdot c}{6}$$

$$\frac{\partial z}{\partial x}=\frac{1}{y}\cdot\frac{-1}{x^2}=\frac{-1}{x^2y} \Rightarrow z'_x(\rho_1,\rho_2)=\frac{-1}{\rho_1^2\rho_2}$$

$$\frac{\partial z}{\partial y}=\frac{1}{x}\cdot\frac{-1}{y^2}=\frac{-1}{xy^2} \Rightarrow z'_y(\rho_1,\rho_2)=\frac{-1}{\rho_1\rho_2^2}$$

$$\rho_3=f(\rho_1,\rho_2)=\frac{1}{\rho_1\cdot\rho_2}$$

$$z-\frac{1}{\rho_1\cdot\rho_2}=\frac{-1}{\rho_1^2\rho_2}(x-\rho_1)+\frac{-1}{\rho_1\rho_2^2}(y-\rho_2) \quad /\cdot\rho_1^2\rho_2^2$$

$$\rho_1^2\rho_2^2 z-\rho_1\rho_2=-\rho_2(x-\rho_1)-\rho_1(y-\rho_2)$$

$$\rho_1^2\rho_2^2 z+\rho_2 x+\rho_2 y=\rho_1\rho_2+\rho_1\rho_2+\rho_1\rho_2 \quad /\cdot\frac{1}{\rho_1\rho_2}$$

$$\frac{x}{\rho_1}+\frac{y}{\rho_1}+\rho_1\rho_2 z=3 \quad /\cdot\frac{1}{3}$$

$$\frac{x}{3\rho_1}+\frac{y}{3\rho_2}+\frac{z}{\frac{3}{\rho_1\rho_2}}=1 \Rightarrow V_{piramide}=\frac{\cancel{3}^{1}\rho_1\cdot 3\rho_2\cdot\frac{3}{\rho_1\rho_2}}{\cancel{6}_{2}}=\frac{9}{2}$$

zapremina piramide za sve tangentne ravni na površ

(#) Nađite udaljenost ishodišta koordinatnog sistema od tangentne ravni (helikoida) $y = x\, tg\frac{z}{a}$ u tački $\left(a, a, \frac{\pi a}{4}\right)$.

Rj: $F'_x(p_1,p_2,p_3)(x-p_1) + F'_y(p_1,p_2,p_3)(y-p_2) + F'_z(p_1,p_2,p_3)(z-p_3) = 0$

jednačina tangentne ravni na površ $F(x,y,z)=0$

$$y - x\, tg\frac{z}{a} = 0$$

$$\frac{\partial F}{\partial x} = -tg\frac{z}{a} \quad \Rightarrow \quad F'_x\left(a,a,\frac{\pi a}{4}\right) = -tg\frac{\frac{\pi a}{4}}{a} = -tg\frac{\pi}{4} = -1$$

$$\frac{\partial F}{\partial y} = 1 \quad \Rightarrow \quad F'_y\left(a,a,\frac{\pi a}{4}\right) = 1$$

$$\frac{\partial F}{\partial z} = \frac{-x}{\cos^2\frac{z}{a}} \cdot \frac{1}{a} = \frac{-x}{a\cos^2\frac{z}{a}} \quad \Rightarrow \quad F'_z\left(a,a,\frac{\pi a}{4}\right) = \frac{-a}{a\cos^2\frac{\pi}{4}} = \frac{-1}{\left(\frac{\sqrt{2}}{2}\right)^2}$$

$$F'_z\left(a,a,\frac{\pi a}{4}\right) = -2$$

$$-1(x-a) + 1\cdot(y-a) + (-2)\left(z - \frac{\pi a}{4}\right) = 0$$

$$-x + y - 2z + a - a + \frac{\pi a}{2} = 0$$

$$-x + y - 2z + \frac{\pi a}{2} = 0$$

jednačina tangentne ravni helikoida u tački $\left(a, a, \frac{\pi a}{4}\right)$.

$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\pm\sqrt{A^2 + B^2 + C^2}}, \quad O(0,0,0)$$

$$d = \frac{0 + 0 + 0 + \frac{\pi a}{2}}{\sqrt{1+1+4}} = \frac{\pi a}{2\sqrt{6}}$$

udaljenost početka koordinatnog sistema od tangentne ravni

(#) Napisati jednačinu tangentne ravni i normale na površ $2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$ u tački $M(2,2,1)$.

Rj. Ako površ S ima jednačinu u implicitnom obliku $F(x,y,z)=0$ tada jednačina tangentne ravni i normale na površ S u tački $M(p_1,p_2,p_3)$ se računaju po formuli:

$$\alpha: F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)=0$$

$$n: \frac{x-p_1}{F'_x(p_1,p_2,p_3)} = \frac{y-p_2}{F'_y(p_1,p_2,p_3)} = \frac{z-p_3}{F'_z(p_1,p_2,p_3)}$$

$$2^{\frac{x}{z}} + 2^{\frac{y}{z}} = 8$$

$$\left(\frac{x}{z}\right)'_z = (xz^{-1})'_z = (-1)xz^{-2}$$

$$F(x,y,z) = 2^{\frac{x}{z}} + 2^{\frac{y}{z}} - 8 = 0$$

$$F'_x = 2^{\frac{x}{z}} \ln 2 \cdot \frac{1}{z} \quad \Rightarrow F'_x(2,2,1) = 4\ln 2.$$

$$F'_y = 2^{\frac{y}{z}} \ln 2 \cdot \frac{1}{z} \quad \Rightarrow F'_y(2,2,1) = 4\ln 2$$

$$F'_z = 2^{\frac{x}{z}} \ln 2 \cdot \left(\frac{x}{z}\right)'_z + 2^{\frac{y}{z}} \ln 2 \left(\frac{y}{z}\right)'_z = -\frac{x}{z^2} 2^{\frac{x}{z}} \ln 2 - \frac{y}{z^2} 2^{\frac{y}{z}} \ln 2$$

$$= -\frac{1}{z^2} \ln 2 \left(x 2^{\frac{x}{z}} + y 2^{\frac{y}{z}}\right)$$

$$F'_z(2,2,1) = -\ln 2 (2\cdot 4 + 2\cdot 4) = -16\ln 2$$

$$4\ln 2 (x-2) + 4\ln 2 (y-2) + (-16\ln 2)(z-1) = 0.$$

$$4x\ln 2 + 4y\ln 2 - 16z\ln 2 + 8\ln 2 = 0$$ jednačina tangentne ravni

$$\frac{x-2}{4\ln 2} = \frac{y-2}{4\ln 2} = \frac{z-1}{-16\ln 2} \Rightarrow \frac{x-2}{1} = \frac{y-2}{1} = \frac{z-1}{-4}$$

jednačina normale na površ

(#) Naći jednačinu tangentne ravni elipsoida $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ koja na koordinatnim osama odsjeca jednake pozitivne odsječke.

Rj. Jednačina tangentne ravni na površ $F(x,y,z)=0$ u tački $M(p_1,p_2,p_3)$ ima jednačinu $F'_x(p_1,p_2,p_3)(x-p_1)+F'_y(p_1,p_2,p_3)(y-p_2)+F'_z(p_1,p_2,p_3)(z-p_3)$

Nađimo jednačinu tangentne ravni na elipsoid u proizvoljnoj tački $M(p_1,p_2,p_3)$: (U našem slučaju $F(x,y,z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$)

$$F'_x=\frac{1}{a^2}\cdot 2x=\frac{2x}{a^2}, \quad F'_y=\frac{2y}{b^2}, \quad F'_z=\frac{2z}{c^2}$$

$$F'_x(M)=\frac{2p_1}{a^2}, \quad F'_y(M)=\frac{2p_2}{b^2}, \quad F'_z(M)=\frac{2p_3}{c^2}$$

$$\frac{2p_1}{a^2}(x-p_1)+\frac{2p_2}{b^2}(y-p_2)+\frac{2p_3}{c^2}(z-p_3)=0 \quad /\cdot\frac{1}{2}$$

$$\frac{p_1}{a^2}x+\frac{p_2}{b^2}y+\frac{p_3}{c^2}z=\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}$$

Napišimo jednačinu ravni u kanonskom obliku

$$\frac{x}{\frac{a^2}{p_1}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}+\frac{y}{\frac{b^2}{p_2}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}+\frac{z}{\frac{c^2}{p_3}\left(\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}\right)}=1$$

Odavde možemo primjetiti da ako želimo da jednačina tangentne ravni na kordanatnim osama odsjeca jednake odsječke, potrebno i dovoljno je da $\frac{a^2}{p_1}=\frac{b^2}{p_2}$, $\frac{a^2}{p_1}=\frac{c^2}{p_3}$ i $\frac{b^2}{p_2}=\frac{c^2}{p_3}$. ...(*)

Isto tako primjetimo da je $\frac{p_1^2}{a^2}+\frac{p_2^2}{b^2}+\frac{p_3^2}{c^2}=1$ ...(**) (ZAŠTO?)

$$(*) \Rightarrow p_1=\frac{a^2}{b^2}p_2, \quad p_3=\frac{c^2}{b^2}p_2 \quad ...(1)$$

Sad imamo:

$$\frac{x}{\frac{a^2}{\frac{a^2}{b^2}p_2}}+\frac{y}{\frac{b^2}{p_2}}+\frac{z}{\frac{c^2}{\frac{c^2}{b^2}p_2}}=1 \quad /:p_2$$

$$\frac{x}{b^2}+\frac{y}{b^2}+\frac{z}{b^2}=\frac{1}{p_2}$$

Kada (1) stavimo u (**) dobićemo da je $p_2=\frac{b^2}{\sqrt{a^2+b^2+c^2}}$

Prema tome:

$x+y+z=\sqrt{a^2+b^2+c^2}$ je jednačina tražene tangente

Zadaci za vježbu

Površi

U zadacima 3410 — 3419 sastaviti jednčine tangencijalnih ravni i normala za date površi u navedenim tačkama.

3410. $z=2x^2-4y^2$ u tački $(2,\ 1,\ 4)$.

3411. $z=xy$ u tački $(1,\ 1,\ 1)$

3412. $z=\dfrac{x^3-3axy+y^3}{a^2}$ u tački $(a,\ a,\ -a$

3413. $z=\sqrt{x^2+y^2}-xy$ u tački $(3,\ 4,\ -7)$.

3414. $z=\operatorname{arctg}\dfrac{y}{x}$ u tački $\left(1,\ 1,\ \dfrac{\pi}{4}\right)$.

3415. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ u tački $\left(\dfrac{a\sqrt{3}}{3},\ \dfrac{b\sqrt{3}}{3},\ \dfrac{c\sqrt{3}}{3}\right)$.

3416. $x^3+y^3+z^3+xyz-6=0$ u tački $(1,\ 2,\ -1)$.

3417. $3x^4-4y^3z+4z^2xy-4z^3x+1=0$ u tački $(1,\ 1,\ 1)$.

3418. $(z^2-x^2)\,xyz-y^5=5$ u tački $(1,\ 1,\ 2)$.

3419. $4+\sqrt{x^2+y^2+z^2}=x+y+z$ u tački $(2,\ 3,\ 6)$.

3420. pokazati da jednačina tangencijalne ravni elipsoida $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ u proizvoljnoj tački $M_0(x_0,\ y_0,\ z_0)$ glasi:

$$\frac{x_0x}{a^2}+\frac{y_0y}{b^2}+\frac{z_0z}{c^2}=1.$$

3421. Naći tangencijalnu ravan elipsoida $x^2+2y^2+z^2=1$ paralelnu ravni $x-y+2z=0$.

3422. Naći tangencijalnu ravan elipsoida $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ koja od koordinatnih osa odseca jednake pozitivne odsečke.

3423. Pokazati da se površi $x+2y-\ln z+4=0$ i $x^2-xy-8x+z+5=0$ dodiruju (tj. imaju zajedničku tangencijalnu ravan) u tački $(2.\ -3,\ 1)$.

3424. Dokazati da se sve tangencijalne ravni površi $z = xf\left(\frac{y}{x}\right)$ seku u jednoj tački.

3425. Sastaviti jednačine tangencijalne ravni i normale sfere $\boldsymbol{r}\{u\cos v, u\sin v, \sqrt{a^2-u^2}\}$ u tački $\boldsymbol{r}_0\{x_0, y_0, z_0\}$.

3426. Sastaviti jednačine tangencijalne ravni i normale hiperboličnog paraboloida $\boldsymbol{r}\{a(u+v), b(u-v), uv\}$ u proizvoljnoj tački $\boldsymbol{r}_0\{x_0, y_0, z_0\}$.

3427. Dokazati da su sfere $x^2+y^2+z^2=ax$ i $x^2+y^2+z^2=by$ uzajamno normalne.

3428. Pokazati da tangencijalne ravni površi $xyz=a^3$ u svakoj njenoj tački obrazuju sa koordinatnim ravnima tetraedre konstantne zapremine i naći tu zapreminu.

3429. Pokazati da tangencijalne ravni površi $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$ odsecaju od koordinatnih osa odsečke čiji zbir ima vrednost a.

3430. Za površ $z=xy$ sastaviti jednačinu tangencijalne ravni, normalne na pravu

$$\frac{x+2}{2}=\frac{y+2}{1}=\frac{z-1}{-1}.$$

3431. Pokazati da je za površ $x^2+y^2+z^2=y$ dužina odsečka normale između površi i ravni xOy jednaka rastojanju od koordinatnog početka do prodora normale kroz tu ravan.

3432. Dokazati da normala obrtnog elipsoida $\frac{x^2+z^2}{9}+\frac{y^2}{25}=1$ u svakoj njegovoj tački $P(x, y, z)$ zaklapa jednake uglove sa pravama PA i PB ako je $A(0, -4, 0)$ i $B(0, 4, 0)$.

3433. Dokazati da sve normale obrtne površi $z=f(\sqrt{x^2+y^2})$ presecaju osu obrtanja.

3434. Za površ $x^2-y^2-3z=0$ naći tangencijalnu ravan koja prolazi kroz tačku $A(0, 0, -1)$ i paralelna je pravoj $\frac{x}{2}=\frac{y}{1}=\frac{z}{2}$.

3435. Na sferi $x^2+y^2+z^2-6y+4z=12$ naći tačke u kojima su tangencijalne ravni paralelne koordinatnim ravnima.

3436. Naći tangencijalnu ravan površi $x=u+v$, $y=u^2+v^2$, $z=u^3+v^3$ u proizvoljnoj tački:

a) uzimajući jednačine površi u parametarskom vidu;
b) napisavši jednačinu ove površi u obliku $z=f(x, y)$.

3437. Naći geometrijsko mesto podnožja normala povučenih iz koordinatnog početka na tangencijalne ravni obrtnog paraboloida $2pz=x^2+y^2$.

3438. Naći geometrijsko mesto podnožja normala spuštenih iz koordinatnog početka na tangencijalne ravni površi $xyz=a^3$.

Rješenja

3410. $8x-8y-z=4$; $\frac{x-2}{8}=\frac{y-1}{-8}=\frac{z-4}{-1}$.

3411. $x+y-z-1=0$; $\frac{x-1}{1}=\frac{y-1}{1}=\frac{z-1}{-1}$.

3412. $z+a=0$, $x=a$, $y=a$.

3413. $17x+11y+5z=60$; $\frac{x-3}{17}=\frac{y-4}{11}=\frac{z+7}{5}$.

3414. $x-y+2z-\frac{\pi}{2}=0$; $\frac{x-1}{1}=\frac{y-1}{-1}=\frac{z-\frac{\pi}{2}}{2}$.

3415. $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=\sqrt{3}$;

$$a\left(x-\frac{a\sqrt{3}}{3}\right)=b\left(y-\frac{b\sqrt{3}}{3}\right)=c\left(z-\frac{c\sqrt{3}}{3}\right)$$

3416. $x+11y+5z-18=0$; $\frac{x-1}{1}=\frac{y-2}{11}=\frac{z+1}{5}$.

3417. $3x-2y-2z+1=0$; $\frac{x-1}{3}=\frac{y-1}{-2}=\frac{z-1}{-2}$.

3418. $2x+y+11z-25=0$; $\frac{x-1}{2}=\frac{y-1}{1}=\frac{z-2}{11}$.

3419. $5x+4y+z-28=0$; $\frac{x-2}{5}=\frac{y-3}{4}=\frac{z-6}{1}$.

3421. $x-y+2z=\sqrt{\frac{11}{2}}$ i $x-y+2z=-\sqrt{\frac{11}{2}}$.

3422. $x+y+z=\sqrt{a^2+b^2+c^2}$.

3424. Sve ravni prolaze kroz koordinatni početak.

3425. $x_0x+y_0y+z_0z=a^2$; $\frac{x}{x_0}=\frac{y}{y_0}=\frac{z}{z_0}$.

3426. $\frac{xx_0}{a^2}-\frac{yy_0}{b^2}=2(z+z_0)$; $\frac{a(x-x_0)}{bx_0}=-\frac{b(y-y_0)}{ay_0}=\frac{z-z_0}{-2ab}$.

3428. $\frac{9}{2}a^3$. 3430. $2x+y-z=2$. 3434. $4x-2y-3z=3$.

3435. Paralelna ravni xOy u tačkama (0, 3, 3) i (0 3, —7); ravni yOz u tačkama (5, 3, —2) i (—5, 3, —2); ravni xOz u tačkama (0, —2, —2) i (0, 8, —2).

3436. a) $6u_0v_0x-3(u_0+v_0)y+2z+(u_0+v_0)(u_0^2-4u_0v_0+v_0^2)=0$;

b) $3(x_0^2-y_0)x-3x_0(y+y_0)+2z+4z_0=0$.

3437. $2z(x^2+y^2+z^2)+p(x^2+y^2)=0$. 3438. $(x^2+y^2+z^2)^3=27a^3xyz$.